

Expansions about the gamma for the distribution and quantiles of a standard estimate

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Abstract: We give expansions for the distribution, density, and quantiles of an estimate, building on results of Cornish, Fisher, Hill, Davis and the authors. The estimate is assumed to be non-lattice with the standard expansions for its cumulants. By expanding about a skew variable with matched skewness, one can drastically reduce the number of terms needed for a given level of accuracy. The building blocks generalize the Hermite polynomials. We demonstrate with expansions about the gamma.

Keywords: Bell polynomials; Gamma distribution; Normal distribution.

1 Introduction and summary

The Cornish Fisher expansions due to Cornish and Fisher (1937) and Hill and Davis (1968) have received applications in many areas of statistics. The Cornish Fisher expansions also have applications in many applied areas, including risk measures for hedge funds, margin setting of index futures, structural equation models, modified sudden death tests, blind inversion of Wiener systems, GPS positioning accuracy estimation, steady-state simulation analysis, blind separation of post-nonlinear mixtures, cycle time quantile estimation, estimation of the maximum average time to flower, performance of Skart, testing and evaluation, load flow in systems with wind generation, Value-at-Risk portfolio optimization, quantile mechanics, channel capacity in communications theory, economics, financial intermediation and physics. Three of the most recent papers applying Cornish Fisher expansions to these areas are: Alfredo and Arunachalam (2011), Simonato (2011) and Zhang *et al.* (2011).

The aim of this paper is to develop technical tools so that the Cornish Fisher expansions could have wider applications. In particular, we show how one can drastically reduce the number of terms needed for a given level of accuracy. The building blocks involve Bell polynomials and Hermite polynomials. In-built routines for these polynomials are available in most computer algebra packages.

Let θ be an unknown real parameter with a non-lattice estimate $\hat{\theta}$ having the standard

asymptotic cumulant expansions

$$\kappa_r(\hat{\theta}) \approx \sum_{i=r-1}^{\infty} a_{ri\theta} n^{-i}, \quad r \geq 1, \quad (1.1)$$

where $a_{10\theta} = \theta$, each $a_{ri\theta}$ is bounded in n , and $a_{21\theta}$ is bounded away from zero as n increases. We call such an estimate a *standard estimate*. For example, a smooth function of a sample mean is a standard estimate - see Withers (1983). Formulas for the leading coefficients were given for parametric estimates in Withers (1982) and for non-parametric estimates in Withers (1983, 1988). The standardized form

$$Y_{01\theta} = (n/a_{21\theta})^{1/2} (\hat{\theta} - \theta) \quad (1.2)$$

has cumulants expandable as

$$\kappa_r(Y_{01\theta}) \approx n^{r/2} \sum_{i=r-1}^{\infty} A_{ri\theta} n^{-i}, \quad r \geq 1, \quad (1.3)$$

where

$$A_{10\theta} = 0, \quad A_{ri\theta} = a_{ri\theta}/a_{21\theta}^{r/2} \text{ for } (r, i) \neq (1, 0). \quad (1.4)$$

Set $P_n(x) = \Pr(Y \leq x)$ for $Y = Y_{01\theta}$, and denote its density by $p_n(x)$. Cornish and Fisher (1937), and Fisher and Cornish (1960) gave expansions for $P_n(x)$ and its inverse. These can be re-written in the form

$$P_n(x) \approx P(x) - p(x) \sum_{r=1}^{\infty} n^{-r/2} h_r(x, L), \quad (1.5)$$

$$P^{-1}(P_n(x)) \approx x - \sum_{r=1}^{\infty} n^{-r/2} f_r(x, L), \quad (1.6)$$

$$P_n^{-1}(P(x)) \approx x + \sum_{r=1}^{\infty} n^{-r/2} g_r(x, L), \quad (1.7)$$

where $P = \Phi$ and $p = \phi$ are the distribution and density of a standard normal variable, denoted $N \sim \mathcal{N}(0, 1)$, $e_r(x, L)$ is a polynomial in both x and $L = (L_1, L_2, \dots)$ for $e = h, f, g$,

$$L_r = l_r/r!, \quad l_r \approx \sum_{j=0}^{\infty} A_{r,r+j-\delta,\theta} n^{-j}, \quad \delta = I(r \geq 3), \quad (1.8)$$

and $I(\cdot)$ is the indicator function. However, they gave no indication of how to truncate these expansions for each adjusted cumulant l_r . This was remedied in Withers (1984) which showed:

Theorem 1.1 *With notation as above,*

$$P_n(x) \approx P(x) - p(x) \sum_{r=1}^{\infty} n^{-r/2} h_r(x), \quad (1.9)$$

$$P^{-1}(P_n(x)) \approx x - \sum_{r=1}^{\infty} n^{-r/2} f_r(x), \quad (1.10)$$

$$P_n^{-1}(P(x)) \approx x + \sum_{r=1}^{\infty} n^{-r/2} g_r(x) \quad (1.11)$$

for $e = h, f, g$, where $e_r(x)$ is a polynomial in both x and $A = \{A_{ri}\}$, $A_{ri} = A_{ri\theta}$, and again, $P = \Phi$, $p = \phi$. The cumulant coefficients needed for $e_r(x)$, ($e = h, f, g$) are as follows:

$$\begin{aligned}
&\text{for } r = 1 : A_{11} \quad A_{32}; \\
&\text{for } r = 2 : A_{22} \quad A_{43}; \\
&\text{for } r = 3 : A_{12} \quad A_{33} \quad A_{54}; \\
&\text{for } r = 4 : A_{23} \quad A_{44} \quad A_{65}; \\
&\text{for } r = 5 : A_{13} \quad A_{34} \quad A_{55} \quad A_{76}; \\
&\text{for } r = 6 : A_{24} \quad A_{45} \quad A_{66} \quad A_{87}.
\end{aligned}$$

Proof: The expansions (1.9)-(1.11) are obtained by substituting l_r of (1.8) into $e_r(x, L)$ of (1.5)-(1.7), giving

$$e_r(x, L) \approx \sum_{i=0}^{\infty} e_{ri}(x) n^{-i} \text{ say,} \quad (1.12)$$

$$e_r(x) = \sum_{0 \leq i < r/2} e_{r-2i,i}(x) = e_r(x, L_0) + \Delta_{re} \text{ say,} \quad (1.13)$$

where

$$e_{r0}(x) = e_r(x, L_0), \quad \Delta_{re} = \sum_{1 \leq i < r/2} e_{r-2i,i}(x), \quad (1.14)$$

$$L_0 = (L_{01}, L_{02}, \dots), \quad L_{0r} = l_{0r}/r!, \quad l_{0r} = A_{r,r-\delta,\theta}, \quad (1.15)$$

the leading term of l_r . \square

To apply the expansions of Theorem 1.1, one can calculate $\Phi(x)$ and its inverse using say NAG routines G01EAF and G01FAF: see http://www.nag.co.uk/numeric/numerical_libraries.asp (Column 1 of Table II of Cornish and Fisher (1937, page 318) gave the main quantiles of N to nine decimal places.) By (1.11), $\hat{\theta}$ has $P(x)$ -quantile

$$\theta + (a_{21\theta}/n)^{1/2} \sum_{r=0}^{\infty} n^{-r/2} g_r(x), \quad (1.16)$$

where $g_0(x) = x$. A drawback of the method is the increasingly large number of terms in each $e_r(x)$ as r increases. The chief contributor is the skewness coefficient $l_{03} = A_{32}$, followed by the bias coefficient $l_{01} = A_{11}$ and the second order variance coefficient $l_{02} = A_{22}$. We now show how to remove these terms. Except for sample means, $\mathbb{E} \hat{\theta}$ and $\text{var}(\hat{\theta})$ are unknown, but they can be approximated by truncating their cumulant expansions. Set

$$s_{rJ\theta} = \sum_{i=r-1}^J a_{ri\theta} n^{-i}, \quad J \geq r-1, \quad (1.17)$$

$$Y_{JK\theta} = s_{2K\theta}^{-1/2} \left(\hat{\theta} - s_{1J\theta} \right), \quad J \geq 0, K \geq 1. \quad (1.18)$$

Then,

$$\kappa_r(Y_{JK\theta}) \approx n^{r/2} \sum_{i=r-1}^{\infty} A_{ri\theta}^{JK} n^{-i}, \quad r \geq 1, \quad (1.19)$$

where $A_{1i\theta}^{JK} = 0$ for $i \leq J$, $A_{2i\theta}^{JK} = 0$ for $i \leq K$, and the other $A_{ri\theta}^{JK}$ are given in terms of $\{A_{ri\theta}\}$ of (1.4) in Section 6. (For $r \geq 2$, $A_{ri\theta}^{JK}$ does not depend on J .) Suppose that $a_{32\theta} > 0$. (If $a_{32\theta} < 0$, replace $(\hat{\theta}, \theta)$ by $(-\hat{\theta}, -\theta)$.)

Now let \hat{w} be another non-lattice standard estimate with the standard cumulant expansion:

$$\kappa_r(\hat{w}) \approx \sum_{i=r-1}^{\infty} a_{riw} m^{-i}, \quad r \geq 1,$$

where $a_{10w} = w$. Since m is arbitrary up to a multiplier, we can take $m = n\tau$ for some constant $\tau > 0$. We also assume that $a_{32w} > 0$. (If $a_{32w} < 0$, replace (\hat{w}, w) by $(-\hat{w}, -w)$.) By Theorem 6.1 below,

$$\kappa_r(Y_{JK\theta}) - \kappa_r(Y_{JKw}) \approx n^{r/2} \sum_{i=r-1}^{\infty} A_{ri}^{JK} n^{-i},$$

where

$$A_{ri}^{JK} = A_{ri\theta}^{JK} - \tilde{A}_{riw}^{JK}, \quad \tilde{A}_{riw}^{JK} = \tau^{r/2-i} A_{riw}^{JK}. \quad (1.20)$$

By Withers and Nadarajah (2012), the expansions (1.5)-(1.7), (1.9)-(1.11) remain true for

$$P_n(x) = \Pr(Y \leq x), \quad P(x) = \Pr(X \leq x), \quad (1.21)$$

and $p_X(x) = p(x)$ is the density of X , if

$$\kappa_r(Y) - \kappa_r(X) \approx n^{r/2} \sum_{i=r-1}^{\infty} A_{ri} n^{-i}, \quad (1.22)$$

and X, Y are non-lattice. So, these expansions hold for

$$Y = Y_{JK\theta}, \quad X = Y_{JKw}, \quad A_{ri} = A_{ri}^{JK}. \quad (1.23)$$

We assume that w and the cumulant coefficients a_{riw} are known. To apply these expansions, we need to be able to calculate $P(x)$ and its inverse accurately. For X linear in gamma, this can be done using NAG routines G01EFF and G01FFF.

Now choose τ so that $A_{32}^{JK} = 0$, that is,

$$\tau = (A_{32w}/A_{32\theta})^2. \quad (1.24)$$

This has the effect of roughly halving the number of terms in each $e_r(x)$. Table 1.1 compares the number of terms in $e_r(x)$ for different choices of J, K . The number is written as $N + M$, where N is the number of terms in $e_r(x, L_0)$ and M is the number of terms in $\Delta_{re}(x)$ of (1.14). For example, the line e_0 refers to the approximation

$$(n/a_{21\theta})^{1/2} (\hat{\theta} - \theta) \stackrel{\mathcal{L}}{=} (m/a_{21w})^{1/2} (\hat{w} - w), \quad m = n\tau,$$

with τ of (1.24). The four columns after the columns headed J and K , accumulate the number of terms needed. That is, they give the number of terms needed for error $O(n^{-(r+1)/2})$ when (1.9)-(1.11) are truncated after $r + 1$ terms. The final column gives the percentage

savings in number of terms over the Cornish-Fisher expansion, which amounts to choosing $\hat{w} \sim \mathcal{N}(0, n^{-1})$.

For example, to calculate a quantile of $\hat{\theta}$ to $O(n^{-7/2})$ via P_n^{-1} of (1.11), the Cornish-Fisher method requires calculating $48+29=77$ terms, but using $(J=3, K=4)$ requires only $11+7=18$ terms - a saving of 77 percent. Similarly, to calculate a p -value of $\hat{\theta}$ to $O(n^{-2})$ via P_n of (1.10), the Cornish-Fisher method requires calculating $14+2=16$ terms, but using $J=K=2$ requires only $3+1=4$ terms - a saving of 75 percent. The percentage savings increases with r .

[Table 1.1 about here.]

We draw attention to the choice $J=K=1$, since then $e_1(x)=0$, and the gamma approximation has error only $O(n^{-1})$, not the usual $O(n^{-1/2})$. This is the simple approximation

$$Y_{11\theta} = (n/a_{21\theta})^{1/2} \left(\hat{\theta} - \theta - a_{11\theta} n^{-1} \right) \stackrel{\mathcal{L}}{=} (m/a_{21w})^{1/2} (\hat{w} - w - a_{11w} m^{-1}), \quad m = n\tau$$

with τ of (1.24).

Increasing J or K beyond those given in the table, does not decrease the number of terms in $e_r(x)$. However, if we want to calculate $e_r(x)$ up to say $r=6$, then we should choose J, K as given on the line for $r=6$, that is, $J=3, K=4$.

We now give $e_r(x)$ up to $r=6$ in terms of *the generalized Hermite polynomial*,

$$H_r = H_{rX}(x) = p(x)^{-1} (-D)^r p(x), \quad D = d/dx, \quad r \geq 0. \quad (1.25)$$

$H_r = H_{rX}(x)$ may not be a polynomial in x , but it *is* a polynomial in $\mathbf{a} = (a_1, a_2, \dots)$, where

$$a_r = a_{rX} = D^r a(x), \quad a(x) = a_X(x) = -\ln p(x), \quad r \geq 1, \quad (1.26)$$

with generating function

$$A_X(x, t) = \sum_{r=0}^{\infty} a_{rX} t^r / r! = a_X(x+t) = -\ln p_X(x+t). \quad (1.27)$$

So, $a_1 = H_1$ and $a(x)$ is convex if and only if $a_2 \geq 0$. The function a_r is simpler to compute than H_r , (only the first two are non-zero for the normal), but expressions in terms of \mathbf{a} are longer (often much longer) than expressions in terms of \mathbf{H} . Also H_r is easily found from its generating function

$$B_X(x, t) = \sum_{r=0}^{\infty} H_{rX}(x) t^r / r! = p_X(x-t) / p_X(x). \quad (1.28)$$

For $e = h, f, g$, Theorem 1.2 gives the much simpler form for $e_r(x)$ of (1.13).

Theorem 1.2 *With notation as above,*

$$e_r(x) = \nabla_r + \nabla_{re}$$

for $e = h, f, g$, where

$$\begin{aligned}\nabla_r &= \sum_{(r+1)/2 \leq i \leq r+1} \bar{A}_{2i-r,i} H_{2i-r-1}, \\ \bar{A}_{ri} &= A_{ri}/r!. \end{aligned} \tag{1.29}$$

In particular,

$$\begin{aligned}\nabla_1 &= 0 \text{ if } J \geq 1, \\ \nabla_2 &= \bar{A}_{43} H_2 \text{ if } K \geq 2, \\ \nabla_3 &= \bar{A}_{33} H_2 + \bar{A}_{54} H_4 \text{ if } J \geq 2, \\ \nabla_4 &= \bar{A}_{44} H_3 + \bar{A}_{65} H_5 \text{ if } K \geq 3, \\ \nabla_5 &= \bar{A}_{34} H_2 + \bar{A}_{55} H_4 + \bar{A}_{76} H_6 \text{ if } J \geq 3, \\ \nabla_6 &= \bar{A}_{45} H_3 + \bar{A}_{66} H_5 + \bar{A}_{87} H_7 \text{ if } K \geq 4.\end{aligned}$$

Also,

$$\begin{aligned}\nabla_{re} &= 0 \text{ if } r \leq 3, \quad \nabla_{4e} = [4^2]_0 e(4^2), \\ \nabla_{5e} &= [45]_0 e(45) + [34]_1 e(34), \\ \nabla_{6e} &= \sum \{[\pi]_0 e(\pi) : \pi = 5^2, 46, 4^3\} + \sum \{[\pi]_1 e(\pi) : \pi = 4^2, 35\} + [3^2]_2 e(3^2),\end{aligned}$$

where the $e(\pi)$, $[\pi]_i$ needed for $e_r(x)$ are as follows: Firstly, there is the special case

$$h(ij \cdots) = H_{i+j+\cdots-1} \text{ so that } h(1^{i_1} 2^{i_2} \cdots) = H_{1i_1+2i_2+\cdots-1}$$

by (1.37), where $H_{k \cdot r} = D^r H_k$. The other expressions needed for $e_r(x)$ are:

$$\begin{aligned}\text{For } r = 4 : \quad [4^2]_0 &= \bar{A}_{43}^2/2!, \\ f(4^2) &= H_7 - H_1 H_3^2, \quad g(4^2) = H_7 - 2H_3 H_4 + H_1 H_3^2. \\ \text{For } r = 5 : \quad [45]_0 &= \bar{A}_{43} \bar{A}_{54}, \\ f(45) &= H_8 - H_1 H_3 H_4, \quad g(45) = H_8 - H_3 H_5 - H_4^2 + H_1 H_3 H_4, \\ [34]_1 &= \bar{A}_{33} \bar{A}_{43}, \quad f(34) = H_6 - H_1 H_2 H_3, \quad g(34) = H_6 - H_2 H_4 - H_3^2 + H_1 H_2 H_3. \\ \text{For } r = 6 : \quad [5^2]_0 &= \bar{A}_{54}^2/2!, \\ f(5^2) &= H_9 - H_1 H_4^2, \quad g(5^2) = H_9 - 2H_4 H_5 + H_1 H_4^2, \\ [46]_0 &= \bar{A}_{43} \bar{A}_{65}, \\ f(46) &= H_9 - H_1 H_3 H_5, \quad g(46) = H_9 - H_3 H_6 - H_4 H_5 + H_1 H_3 H_5, \\ [4^3]_0 &= \bar{A}_{43}^3/3!, \quad f(4^3) = H_{11} - 3H_1 H_3 H_7 - H_2 H_3^3 + 3H_1^2 H_3^3, \\ g(4^3) &= H_{11} - 3H_3 H_8 - 3H_4 H_7 + 3H_1 H_3 H_7 + 3H_3^2 H_5 \\ &+ 6H_3 H_4^2 - 9H_1 H_3^2 H_4 - H_2 H_3^3 + 3H_1^2 H_3^3, \\ [4^2]_1 &= \bar{A}_{43} \bar{A}_{44}, \\ [35]_1 &= \bar{A}_{33} \bar{A}_{54}, \quad f(35) = H_7 - H_1 H_2 H_4, \quad g(35) = H_7 - H_3 H_4 - H_2 H_5, \\ [3^2]_2 &= \bar{A}_{33}^2/2!, \quad f(3^2) = H_5 - H_1 H_2^2, \quad g(3^2) = H_5 - 2H_2 H_3 + H_1 H_2^2.\end{aligned}$$

Now choose $w = 1$, $\hat{w} = G_m/m$, where G_m is a gamma variable with mean m . By (1.20),

$$\begin{aligned} A_{r,r-1}^{JK} &= A_{r,r-1,\theta} - (r-1)! (A_{32\theta}/2)^{r-2}, \quad r \geq 4, \\ A_{rr}^{JK} &= A_{rr\theta}^{JK} = A_{rr\theta} + d_{r1}A_{r,r-1,\theta}, \quad r \geq 2, K \geq 2, \\ A_{r,r+1}^{JK} &= A_{r,r+1\theta}^{JK} = A_{r,r+1\theta} + d_{r1}A_{rr\theta} + d_{r2}A_{r,r-1,\theta}, \quad r \geq 2, K \geq 3, \end{aligned}$$

where

$$d_{r1} = (-r/2)A_{23\theta}, \quad d_{r2} = (-r/2)A_{23\theta} + \binom{-r/2}{2}A_{22\theta}^2.$$

Sections 2 to 5 deal with the *general* case (1.21)-(1.22). Section 2 gives simple formulas for \mathbf{H} and \mathbf{a} of (1.25) and (1.26) for X a gamma variable with mean m , and so for X of (1.23). Examples in Section 2 include the distribution and quantiles of the sample variance and the Studentized mean for non-normal populations.

Section 3 re-expresses $h_r(x, L)$ using a change of notation that does away with the fractional coefficients in all of the papers referred to above. Fractions are eliminated by giving results, not in terms of (l_1, l_2, \dots) , but in terms of

$$[1^{i_1} 2^{i_2} \dots] = \left(L_1^{i_1}/i_1! \right) \left(L_2^{i_2}/i_2! \right) \dots, \quad (1.30)$$

where $L_r = l_r/r!$. We shall prove the following theorem.

Theorem 1.3 *With notation as above,*

$$h_r(x, L) = \sum_{k=r, r+2, \dots, 3r} C_{rk} H_{k-1}(x), \quad (1.31)$$

where

$$C_{rk} = \sum \{[\pi] : i \in \mathcal{H}_{rk}\},$$

and \mathcal{H}_{rk} is the set of all partitions $\pi = 1^{i_1} \dots k^{i_k}$ of k such that

$$S(1)i_1 + \dots + S(k)i_k = r,$$

where

$$S(r) = rI(r \leq 2) + (r-2)I(r \geq 3).$$

For example,

$$C_{00} = 1, \quad C_{rr} = \sum_{0 \leq i \leq r/2} [1^{r-2i} 2^i]. \quad (1.32)$$

The other C_{rk} are obtained from these using

$$C_{r,r+2i} = \sum_{j=0}^{r-i} C_{jj} b_{r-j,i}(\bar{L}), \quad 1 \leq i \leq r, \quad \bar{L}_i = L_{i+2}, \quad b_{ri}(x) = \hat{B}_{ri}(x)/i! \quad (1.33)$$

and $\hat{B}_{ri}(x)$ is the ordinary Bell polynomial - see Appendix A.

Note that (1.32) and (1.33) provide convenient ways to calculate h_r using MAPLE, say. Section 4 proves the following theorem.

Theorem 1.4 *With notation as above, $f_r(x, L)$ and $g_r(x, L)$ of (1.6)-(1.7) can be expressed in the form*

$$e_r(x, L) = \sum \{[\pi] e(\pi) : \pi \in S_r(e)\}, \quad (1.34)$$

where $S_r(e)$ is a set of partitions π of $r, r+2, \dots, 3r$. The coefficients $f(\pi)$, $g(\pi)$ are polynomials in \mathbf{H} .

Alternatively, $f(\pi)$, $g(\pi)$ can be written as polynomials in \mathbf{a} , as done in Appendix D. $f(\pi)$ are obtained via functions $c_r = c_{rX}(x)$ introduced by Hill and Davis (1968). We express these in terms of \mathbf{H} .

Section 5 gives $f_r(x)$ and $g_r(x)$ of (1.10)-(1.11).

Section 7 extends (1.9) and (1.5) to expansions for the density and other derivatives of $P_n(x)$.

Hill and Davis (1968) gave a different motivation: the distribution of the likelihood ratio has an expansion of the form (1.9), with X chi-square, or equivalently, gamma. Given an expansion of the form (1.9), they derive (1.10) and (1.11) giving f_r , g_r in terms of (h_1, h_2, \dots) . Withers and Nadarajah (2012) simplified their results using Bell polynomials.

Another such example is when X is *symmetric* about zero, such as N , or Student's t . In this case H_r is an even function for r even, so that h_r is an odd function for r odd, and $\bar{Y}_n = |Y_{01\theta}|$ satisfies

$$\Pr(\bar{Y}_n \leq x) = \bar{P}(x) - \bar{p}(x) \sum_{r=1}^{\infty} n^{-r} h_{2r}(x),$$

where $\bar{P}(x) = \Pr(|X| \leq x) = P(x) - P(-x)$ has density $\bar{p}(x) = 2p(x)$.

Appendix B gives the interesting expression for H_r in terms of \mathbf{a} ,

$$H_r(x) = (a_1 - D)^r 1 = (-1)^r B_r(-\mathbf{a}), \quad r \geq 0,$$

where B_r is the complete Bell polynomial, and an inverse formula for a_r in terms of \mathbf{H} . It also gives the derivatives of H_r in terms of \mathbf{H} using the functions

$$b_r = (a_1 + D)^r 1 = B_r(\mathbf{a}), \quad r \geq 0. \quad (1.35)$$

Appendix C gives $f(\pi)$ and $g(\pi)$ needed in (1.34) in terms of \mathbf{H} . Appendix D gives the same but in terms of \mathbf{a} . Appendix E gives $e(\pi)$ for $e = f, g$ when X is a gamma variable. Appendix F specializes to $P(x) = \Phi(x) = \Pr(N \leq x)$, the standard normal distribution, the choice used by Cornish and Fisher (1937) and Fisher and Cornish (1960). We give formulas for some $f(\pi)$, $g(\pi)$ that do not hold when X is non-normal. The generating function is $B_N(x, t) = e^{xt - t^2/2} = \mathbb{E} e^{(x+iN)t}$ so that $H_r = He_r$, the r th Hermite polynomial,

$$He_r(x) = \phi(x)^{-1} (-D)^r \phi(x) = \mathbb{E} (x + iN)^r, \quad (1.36)$$

as noted by Withers (2000). Withers and Nadarajah (2011) also extended the results of Cornish and Fisher (1937) to general X and gave the recurrence relation

$$H_r = J_1 H_{r-1}, \quad r \geq 1, \quad (1.37)$$

where $J_1 = H_1 - D$; its Sections 4 and 5 gave H_r for X a standardized gamma or a Student random variable, using ψ_{r+1} for H_r . Its Appendix A gave $e_r(x, L)$ for $r \leq 4$, $e = h, f, g$.

We use the notation

$$\begin{aligned} [\alpha]_j &= \Gamma(\alpha + 1)/\Gamma(\alpha - j + 1) = \alpha(\alpha - 1) \cdots (\alpha - j + 1), \\ (\alpha)_j &= \Gamma(\alpha + j - 1)/\Gamma(\alpha - 1) = \alpha(\alpha + 1) \cdots (\alpha + j - 1). \end{aligned} \quad (1.38)$$

When changing variables, say from Y to $X = (Y - \mu)/\sigma$, it is convenient to set $P_X(x) = \Pr(X \leq x)$, $y = \mu + \sigma x$. Then,

$$\begin{aligned} p_X(x) &= \sigma p_Y(y), \quad a_X(x) = -\ln \sigma + a_Y(y), \\ A_X(x, t) &= -\ln \sigma + A_Y(y, \sigma t), \quad B_X(x, t) = B_Y(y, \sigma t), \\ H_{rX}(x) &= \sigma^r H_{rY}(y), \quad a_{rX}(x) = \sigma^r a_{rY}(y), \quad c_{r+1,X}(x) = \sigma^r c_{rY}(y). \end{aligned} \quad (1.39)$$

2 Expansions about the gamma, with examples

Y_w needs to have a shape parameter if we want to reduce A_{32} to zero. Expansions about X a standardized gamma or χ^2 were given in Section 4 of Withers and Nadarajah (2011). However, it is easier to evaluate first **a** and then **H**. Theorem 2.1 gives explicit formulas for these for gamma random variables.

Theorem 2.1 *Let G be a gamma random variable with mean m and density $y^{m-1}e^{-y}/\Gamma(m)$ on $(0, \infty)$. So,*

$$\kappa_r(G) = (r-1)!m, \quad A_{riw} = (r-1)!m^{1-r/2} \quad (2.1)$$

for $\hat{w} = G/m$, $w = 1$. Set

$$\alpha = m - 1, \quad \bar{y} = -1/y. \quad (2.2)$$

For $X = G$, a , a_r of (1.26), H_r and the generating functions (1.27) and (1.28) are given by

$$\begin{aligned} a_G(y) &= y - \alpha \ln y + \ln \Gamma(\mu), \quad A_G(y, t) - a_G(y) = t - \alpha \ln(1 - \bar{y}t), \\ a_{rG}(y) &= I(r=1) + (r-1)! \alpha \bar{y}^r, \quad r \geq 1, \\ B_G(y, t) &= (1 + \bar{y}t)^\alpha e^t, \\ H_{rG}(y) &= \sum_{j=0}^r \binom{r}{j} [\alpha]_j \bar{y}^j = {}_2F_0(-r, -\alpha, \bar{y}), \end{aligned} \quad (2.3)$$

where ${}_pF_q$ is the generalized hypergeometric distribution (see Section 9.14 of Gradshteyn and Ryzhik (1965)). Also

$$Y_{JKw} = (G - \mu)/\sigma, \quad (2.4)$$

where $\sigma^2 = m/s_{2Kw}$ and $\mu = m - s_{1Jw}(m/s_{2K})^{1/2}$. By (1.39), for $X = Y_{JKw}$,

$$H_r = H_{rX}(x) = \sigma^r H_{rG}(y), \quad a_r = \sigma^r a_{rG}(y) \quad \text{at } y = \mu + \sigma x,$$

$$c_{r+1,G} = r!(1 + r\bar{y}) + \sum_{i=2}^r c_{r+1,i}\bar{y}^i, \quad r \geq 1.$$

In particular,

$$c_{r+1,r} = \prod_{k=1}^r (k\alpha + k - 1),$$

$$c_{42} = \alpha(18\alpha + 7), \quad c_{52} = \alpha(72\alpha + 23), \quad c_{53} = 2\alpha(2\alpha + 1)(24\alpha + 11),$$

$$c_{62} = 2\alpha(600\alpha + 163), \quad c_{63} = 2\alpha(489\alpha^2 + 600\alpha + 101),$$

$$c_{64} = 3\alpha(2\alpha + 1)(100\alpha^2 + 113\alpha + 32).$$

Matching skewness by (1.8) and (2.1) gives

$$A_{11} = A_{11\theta}, \quad A_{22} = A_{22\theta}, \quad A_{r,r-1} = A_{r,r-1,\theta} - \tau^{1-r/2}(r-1)!$$

for $r \geq 3$. So, $A_{32} = 0$ if we choose $m = n\tau$ with $\tau^{1/2} = 2/A_{32\theta}$, in (2.4). Then,

$$l_{0r} = A_{r,r-1} = A_{r,r-1,Y} - (r-1)!(A_{32Y}/2)^{r-2}$$

for $r \geq 4$. In particular, $A_{43} = A_{43Y} - 3A_{32Y}^2/2$, $A_{54} = A_{54Y} - 3A_{32Y}^3$, and $A_{65} = A_{65Y} - 15A_{32Y}^4/2$.

Example 2.1 The F variable is defined by $F_{n_1,n_2} = (\chi_{n_1}^2/n_1)/(\chi_{n_2}^2/n_2)$, where the chi-square variable $\chi_{n_1}^2$ is independent of $\chi_{n_2}^2$. Wishart (1947) gave expansions in powers of n_1^{-1} , n_2^{-1} for the cumulants of $Z = 2^{-1} \ln F_{n_1,n_2}$. Setting $n = n_1 + n_2$ say, and $f_i = n/n_i$, it follows that $\hat{\theta} = Z$ satisfies (1.1) holds with $a_{10\theta} = \theta = 0$ and the non-zero $a_{ri\theta}$ given by

$$a_{rr\theta} = [f_2^r + (-1)^r f_1^r] (r-1)!/2,$$

$$a_{r,2j+r-1,\theta} = 2 \left[f_2^{2j+r-1} + (-1)^r f_1^{2j+r-1} \right] (-4)^{j-1} B_j (2j+r-2)!/(2j)!, \quad (j,r) \neq (0,1).$$

The B_j are given by $B_0 = -1$, $B_1 = 1/6$, $B_2 = 1/30$, $B_3 = 1/42$, $B_4 = 1/30$, $B_5 = 5/66$. (This is not the current notation for the Bernoulli numbers. Apart from B_0 , his B_j is what we call $|B_{2j}|$ today.) So, $a_{21\theta} = (f_2 + f_1)/2$. Now redefine n by $n = 2/\sum_{i=1}^2 n_i^{-1}$, the harmonic mean of n_1, n_2 . (We can do this as n is arbitrary provided that it has magnitude $\min(n_1, n_2)$.) So, now $a_{21\theta} = 1$. Writing $a_{ri} = a_{ri\theta}$, the coefficients needed for $e_r(x)$ are

$$e_1 : a_{11} = (f_2 - f_1)/2, a_{32} = (f_2^2 - f_1^2)/2,$$

$$e_2 : a_{22} = (f_2^2 + f_1^2)/2, a_{43} = f_2^3 + f_1^3,$$

$$e_3 : a_{12} = (f_2^2 - f_1^2)/6, a_{33} = f_2^3 - f_1^3, a_{54} = 3(f_2^4 - f_1^4),$$

$$e_4 : a_{23} = (f_2 + f_1)/3, a_{44} = 3(f_2^4 + f_1^4), a_{65} = 12(f_2^5 + f_1^5),$$

$$e_5 : a_{13} = 0, a_{34} = f_2^4 - f_1^4, a_{55} = 12(f_2^5 - f_1^5), a_{76} = 60(f_2^6 - f_1^6),$$

$$e_6 : a_{24} = 0, a_{45} = 4(f_2^5 + f_1^5), a_{66} = 60(f_2^6 + f_1^6), a_{87} = 360(f_2^7 + f_1^7).$$

Cornish and Fisher (1937, page 319) and Fisher and Cornish (1960, page 216) denote $a_{21\theta}/n$ by $\sigma/2$. They illustrated the quantile expansion (1.16) for $n_1 = 24$, $n_2 = 60$, $P(x) =$

0.95, giving columns 1 to 4 of Table 2.1 using $P = \Phi$, the normal distribution. (They give the exact value as .26534844...)

The picture is less rosy if the degrees of freedom are small. In this case the series must be truncated when divergence begins, giving an upper bound to the accuracy achievable, as illustrated by Table 2.2 for degrees of freedom 5 and 5.

We can write $\chi_n^2/n = G_m/m$, where G_m is a gamma variable with mean $m = 2n$. So, switching to $\gamma_i = 2f_i = n/m_i$ and $\hat{\theta} = \ln(G_{m_1}/m_1) - \ln(G_{m_2}/m_2)$, where G_{m_1} is independent of G_{m_2} , it follows that (1.1) holds with $a_{10\theta} = \theta = 0$, and the non-zero $a_{ri\theta}$ given by

$$a_{rr\theta} = [\gamma_2^r + (-1)^r \gamma_1^r] (r-1)!/2,$$

$$a_{r,2j+r-1,\theta} = (-1)^{j-1} \left[\gamma_2^{2j+r-1} + (-1)^r \gamma_1^{2j+r-1} \right] B_j(2j+r-2)/(2j)!, \quad (j,r) \neq (0,1).$$

For example, the leading coefficients are

$$a_{r,r-1,\theta} = (r-2)! [\gamma_2^{r-1} + (-1)^r \gamma_1^{r-1}], \quad r \geq 2.$$

So, $a_{21\theta} = \gamma_2 + \gamma_1 = 1$ if we redefine n as $n = 1/\sum_{i=1}^2 m_i^{-1} = m_1 m_2 / (m_1 + m_2)$, so that now (1.3) holds with $A_{ri\theta} = a_{ri\theta}$. (γ_i is still given by n/m_i .)

[Tables 2.1 and 2.2 about here.]

We end with two non-parametric examples. Suppose that we have a random sample X_1, \dots, X_n of size n from an unknown distribution F with mean $\mu = \mu(F)$ and finite central moments $\mu_r = \mu_r(F)$. Their empirical estimates are

$$\mu(F_n) = \bar{X} = n^{-1} \sum_{j=1}^n X_j, \quad \mu_r(F_n) = n^{-1} \sum_{j=1}^n (X_j - \bar{X})^r,$$

where F_n is the empirical distribution. Set $\nu_r = \mu_r/\mu_2^{r/2}$. By Withers (1983), for $T(F)$ a smooth functional, the cumulants of $\hat{\theta} = T(F_n)$ have an expansion of the form (1.1) with $\theta = T(F)$ and the leading cumulant coefficients $a_{ri\theta}$ given by Theorem 3.1 there.

Example 2.2 The distribution and quantiles of the sample variance, $\hat{\theta} = \mu_2(F_n)$. (After scaling, this is equivalent to choosing the unbiased estimate $s^2 = n\mu_2(F_n)/(n-1)$.) So, $\theta = \mu_2$, $a_{21\theta} = \mu_4 - \mu_2^2$. The leading $a_{ri} = a_{ri\theta}$ of (1.3) are given by Example 4.2 of Withers (1983) in terms of $\delta = \nu_4 - 1$:

$$\begin{aligned} e_1 : a_{11} &= -\mu_2, \quad a_{32} = \mu_6 - 3\mu_4\mu_2 + 2\mu_2^3 - 6\mu_3^2, \\ e_2 : a_{22} &= 4\mu_2^2 - 2\mu_4, \quad a_{43} = \mu_8 - 4\mu_6\mu_2 + 12\mu_4\mu_2^2 - 3\mu_4^2 - 24\mu_5\mu_3 + 96\mu_3^2\mu_2 - 6\mu_2^4, \\ e_3 : a_{12} &= 0, \quad a_{33} = -3\mu_6 + 21\mu_4\mu_2 - 26\mu_2^3 + 18\mu_3^2, \quad a_{54} = \mu_{10} - 5\mu_8\mu_2 - 40\mu_7\mu_3 \\ &\quad - 10\mu_6\mu_4 + 20\mu_6\mu_2^2 - 30\mu_5^2 + 480\mu_5\mu_3 + 360\mu_4\mu_3^2 + 30\mu_4^2 - 60\mu_4\mu_2^3 - 1560\mu_3^2\mu_2^2 + 24\mu_2^5. \end{aligned}$$

Example 2.3 The distribution and quantiles of the Studentized mean, $Y_{01\theta} = n^{1/2}\hat{\theta}$, where $\hat{\theta} = \mu_2(F_n)^{-1/2}(\bar{X} - \mu) = T_0(F_n)$ say. (For a normal sample, $(1 - n^{-1})^{1/2}Y_{01\theta} \sim t_{n-1}$, but otherwise it is simpler to calculate $A_{ri\theta}$ for $Y_{01\theta}$.) So, $a_{10\theta} = 0$, $a_{21\theta} = 1$, and the other leading $a_{ri} = a_{ri\theta} = A_{ri\theta}$ of (1.3) are given in Example 1.2 of Withers (1989b):

$$\begin{aligned} e_1 : a_{11} &= -\nu_3/2, \quad a_{32} = -2\nu_3, \\ e_2 : a_{22} &= 3 + 7\nu_3^2/2, \quad a_{43} = 12 - 2\nu_4 + 12\nu_3^2, \\ e_3 : a_{12} &= (-25\nu_3 + 6\nu_5 - 15\nu_3\nu_4)/16. \end{aligned}$$

(This reference also gives the leading h_r, f_r, g_r when $X = N$.)

3 $h_r(x, L)$ in terms of sums of partitions

This section can be skipped on a first reading. It proves Theorem 1.3.

Proof of Theorem 1.3 Formulas for \mathcal{H}_{rk} are easily derived from (1.32) and (1.33). Here, we only note that each \mathcal{H}_{rk} is a distinct set of partitions of k , and

$$\cup_r \mathcal{H}_{rk} = \cup_{0 \leq i \leq k/2} \mathcal{H}_{k-2i,k}$$

is the set of all the partitions of k .

When $X = N$, equation (3.1) of Withers (1984) gave the formula

$$h_r(x, L) = \sum L_{r_1} \cdots L_{r_j} H_{r_1+\dots+r_j-1}(x)/j! \quad (3.1)$$

summed over $j \geq 1$, $r_1 \geq 1, \dots, r_j \geq 1$, $S(r_1) + \dots + S(r_j) = r$. Its proof follows from the Charlier differential series - see Withers and Nadarajah (2012), so it remains valid for general X .

Let us rewrite (3.1) in the form

$$h_r(x, L) = \sum_{k=1}^{3r} C_{rk} H_{k-1}(x), \quad C_{rk} = \sum_{j=1}^r C_{rkj}, \quad (3.2)$$

where

$$C_{rkj} = \sum L_{r_1} \cdots L_{r_j} / j!$$

is summed over $r_1 \geq 1, \dots, r_j \geq 1$, $S(r_1) + \dots + S(r_j) = r$.

Now suppose that $\{r_1, \dots, r_j\}$ consists of i_1 1s, i_2 2s, \dots , i_k ks. The number of ways this can arise is the multinomial coefficient, $\binom{j}{i_1 \dots i_k}$. So, we can rewrite C_{rkj} using the [] notation of (1.30) in the form

$$C_{rkj} = \sum [1^{i_1} \cdots k^{i_k}]$$

summed over

$$i_1 + \dots + i_k = j, \quad 1i_1 + \dots + ki_k = k, \quad S(1)i_1 + \dots + S(k)i_k = r.$$

(For π a partition, Hill and Davis denote $[\pi]$ by l_π .) The last constraint can be written $i_1 + 2i_2 + \sum_{a=3}^k (a-2)i_a = r$. So, these three constraints can be written

$$i_3 + \dots + i_k = j - i_1 - i_2, \quad 1i_1 + \dots + ki_k = k, \quad i_1 + 2i_2 + \sum_{a=3}^k (a-2)i_a = r.$$

So,

$$C_{rk} = \sum_{i_1+2i_2 \leq r} [1^{i_1} 2^{i_2}] C_{rki_1i_2}, \quad (3.3)$$

where

$$C_{rki_1i_2} = \sum [3^{i_3} \cdots k^{i_k}]$$

is summed over

$$\sum_{a=3}^k i_a = (k-r)/2 = K \text{ say,}$$

$$\sum_{a=3}^k (a-2)i_a = r - i_1 - 2i_2 = R \text{ say.}$$

By (A.2), $C_{rk i_1 i_2} = b_{RK}(\bar{L})$ at $\bar{L}_i = L_{i+2}$.

By (1.32),

$$C_{11} = [1], \quad C_{22} = [1^2] + [2], \quad C_{33} = [1^3] + [12], \quad C_{44} = [1^4] + [1^2 2] + [2^2],$$

$$C_{55} = [1^5] + [1^3 2] + [12^2], \quad C_{66} = [1^6] + [1^4 2] + [1^2 2^2] + [2^3],$$

and we can write (3.3) as

$$C_{rk} = \sum_{j=0}^r C_{jj} b_{r-j,K}(\bar{L}),$$

giving (1.33), since $\hat{B}_{RK} = 0$ for $R < K$, and $C_{rk} = 0$ if $k-r$ is odd or $k < r$. So,

$$\begin{aligned} h_r(x, L) &= \sum_{k=r, r+2, \dots, 3r} C_{rk} H_{k-1}(x) \\ &= \sum_{i=0}^r [C_{rk} H_{k-1}(x)]_{k=r+2i} = \sum_{j=0}^r [C_{rk} H_{k-1}(x)]_{k=3r-2j}. \end{aligned}$$

The proof is complete. \square

We can rewrite (1.33) as

$$C_{r, 3r-2i} = \sum_{j=0}^i C_{jj} b_{r-j, r-i}, \quad r \geq i \geq 0. \quad (3.4)$$

Using Appendix A, we obtain some special cases of the results in Theorem 1.3.

Corollary 3.1 *With notation as above,*

$$C_{r, r+2} = \sum_{j=0}^{r-1} C_{jj} \alpha_{r-j}, \quad (3.5)$$

$$C_{r, 3r} = b_{rr}(\bar{L}) = [3^r], \quad (3.6)$$

$$C_{r, 3r-2} = [13^{r-1}] + b_{r, r-1}(\bar{L}), \quad b_{r, r-1}(\bar{L}) = [3^{r-2} 4], \quad (3.7)$$

$$C_{r, 3r-4} = C_{22} [3^{r-2}] + [13^{r-3} 4] + b_{r, r-2}(\bar{L}), \quad b_{r, r-2}(\bar{L}) = [3^{r-4} 4^2] + [3^{r-3} 5] \quad (3.8)$$

where $\alpha_r = b_{r1}(\bar{L}) = L_{r+2}$ for $r \geq 1$, and a term with a negative power of three is discarded. For example,

$$\begin{aligned} C_{24} &= L_4 + L_1 L_3 = [4] + [13], \quad C_{26} = [3^2], \\ C_{46} &= L_6 + L_1 L_5 + C_{22} L_4 + C_{33} L_3 = [6] + [15] + [1^2 4] + [24] + [1^3 3] + [123], \\ C_{48} &= [4^2] + [35] + [134] + C_{22} [3^2], \quad C_{22} [3^2] = [1^2 3^2] + [23^2], \\ C_{5, 11} &= C_{22} [3^3] + C_{11} [3^2 4] + [34^2] + [3^2 5] = [1^2 3^3] + [13^2 4] + [23^3] + [34^2] + [3^2 5]. \end{aligned}$$

So, using the $[\cdot]$ notation, all the numerical coefficients of the components of $h_r(x, L)$ are 1:

$$\begin{aligned}
h_1(x, L) &= [1] + [3]H_2, \\
h_2(x, L) &= ([1^2] + [2])H_1 + ([13] + [4])H_3 + [3^2]H_5, \\
h_3(x, L) &= ([1^3] + [12])H_2 + ([1^23] + [23] + [14] + [5])H_4 + ([13^2] + [34])H_6 \\
&\quad + [3^3]H_8, \\
h_4(x, L) &= ([1^4] + [1^22] + [2^2])H_3 + ([1^33] + [1^24] + [123] + [15] + [24] + [6])H_5 \\
&\quad + ([1^23^2] + [134] + [23^2] + [35] + [4^2])H_7 + ([13^3] + [3^24])H_9 + [3^4]H_{11},
\end{aligned}$$

and so on.

However, it is safer to calculate $H_r(x, L)$ in MAPLE using (1.31), (1.32), (1.33), (3.4)-(3.8) to avoid the chance of missing a term.

Note how each C_{rk} sums over a distinct set of partitions of k , and how $\sum_r C_{rk}$ sums over all distinct set of partitions of k . When $X = N$, the 2nd term in $h_r(x, L)$, $C_{r,r+2}H_{r+1}(x)$, also occurs in $f_r(x, L)$ and $g_r(x, L)$.

To convert a term to the form given by Cornish and Fisher (1937), Fisher and Cornish (1960) and Withers (1984), it is only necessary to substitute

$$[1^{i_1}2^{i_2}\dots] = \prod_{k=1} (l_k/k!)^{i_k} / i_k! = (1^{i_1}2^{i_2}\dots) \prod_{k=1} l_k^{i_k} / r!,$$

where $r = \sum k i_k$ and $(1^{i_1}2^{i_2}\dots) = r! / \prod_{k=1} k^{i_k} i_k!$ is the partition function.

4 Expressions for $f_r(x, L)$ and $g_r(x, L)$ in terms of (L, \mathbf{H})

We begin by giving the proof of Theorem 1.4.

Proof of Theorem 1.4 By equation (5.6) of Withers and Nadarajah (2012),

$$f_r(x, L) = \sum_{k=1}^r (-1)^{k-1} c_k b_{rk}(h), \quad (4.1)$$

$$g_r(x, L) = \sum_{k=1}^r (-1)^{k-1} D_k b_{rk}(h) \quad (4.2)$$

for b_{rk} of (1.33), where $h_r = h_r(x, L)$. So, (1.31), (4.1) and (4.2) can be written in the form (1.34), where $S_r(h) = \cup_k \mathcal{H}_{rk}$ is the set of partitions $\pi = (i_1 i_2 \dots)$ such that $S(i_1) + S(i_2) + \dots = r$, $h(\pi) = H_{i_1+i_2+\dots-1}$. If π is a partition of k , then the coefficients $f(\pi)$, $g(\pi)$ are polynomials in $\{H_1, \dots, H_{k-1}\}$. The exception is $h(1) = f(1) = g(1) = H_0 = 1$. \square

Corollary 4.1 gives some particular cases of (4.1)-(4.2) and $f(\cdot)$, $g(\cdot)$, $h(\cdot)$.

Corollary 4.1 *With notation as above,*

$$\begin{aligned}
g_1(x, L) &= f_1(x, L) = h_1(x, L) = [1] + [3]H_2, \\
f_4 &= h_4 - c_2 (h_1 h_3 + h_2^2/2) + c_3 h_1^2 h_2/2 - c_4 h_1^4/4!, \\
g_4 &= h_4 - D_2 (h_1 h_3 + h_2^2/2) + D_3 h_1^2 h_2/2 - D_4 h_1^4/4!.
\end{aligned} \quad (4.3)$$

Here, c_k and D_k are the function and operator introduced by Hill and Davis (1968):

$$\begin{aligned} c_1 &= 1, \quad c_{k+1} = K_k c_k = K_k \cdots K_1 1, \\ D_1 &= 1, \quad D_{k+1} = D_k J_k = J_1 \cdots J_k, \end{aligned}$$

where $K_k = kH_1 + D$ and $J_k = kH_1 - D$. Using the expressions for c_k in terms of \mathbf{a} given in Withers and Nadarajah (2012), we obtain

$$\begin{aligned} c_2 &= H_1, \quad c_3 = 3H_1^2 - H_2, \quad c_4 = 15H_1^3 - 10H_1H_2 + H_3, \\ c_5 &= 105H_1^4 - 105H_1^2H_2 + 15H_1H_3 + 10H_2^2 - H_4, \\ c_6 &= 945H_1^5 - 1260H_1^3H_2 + 210H_1^2H_3 + 280H_1H_2^2 - 35H_2H_3 - 21H_1H_4 + H_5. \end{aligned}$$

The coefficient of H_1^r in c_{r+1} is $\mathbb{E} N^{2r}$. That of $H_1^{r-2}H_2$ is $-(r-1)\mathbb{E} N^{2r}/3$.

Other particular cases are

$$h(k) = f(k) = g(k) = H_{k-1}, \quad f(1^i) = 0 \text{ for } i \geq 2, \quad (4.4)$$

$$f(1, k+1) = H_k - H_1H_{k-1} = -H_{k,1}, \quad g(1i) = 0, \quad (4.5)$$

$$f(1^{i-1}2) = \kappa_i(\mathbf{H}) = \sum_{j=1}^i (-1)^j (j-1)! B_{ij}(\mathbf{H}), \quad i \geq 1. \quad (4.6)$$

The last formula, (4.6), is just the familiar formula for κ_i in terms of the non-central moments, $\mathbf{m} = (m_1, m_2, \dots)$:

$$\begin{aligned} \kappa_1 &= m_1, \quad \kappa_2 = m_2 - m_1^2, \quad \kappa_3 = m_3 - 3m_2m_1 + 2m_1^3, \\ \kappa_4 &= m_4 - 4m_3m_1 - 3m_2^2 + 12m_2m_1^2 - 6m_1^4, \dots \end{aligned}$$

as given by equation (3.42) of Stuart and Ord (1987) up to $i = 10$. The formula

$$\kappa_i = \sum_{j=1}^i (-1)^j (j-1)! B_{ij}(\mathbf{m}),$$

and its inverse formula, $m_i = B_i(\kappa)$, where $\kappa = (\kappa_1, \kappa_2, \dots)$, were given by Withers and Nadarajah (2009), and in equation (2) of Comtet (1974, page 160). For example, $f(1^22) = \kappa_3(H) = H_3 - 3H_1H_2 + 2H_1^3$. By (4.3), $h_1(x, L) = f_1(x, L) = g_1(x, L)$ is given by

$$S_1(f) = S_1(g) = \{1, 3\}, \quad f(1) = H_0 = 1, \quad f(3) = H_2.$$

For $e(\pi)$, $S_r(e)$ needed by (1.34) to compute $e_r(x, L)$, $2 \leq r \leq 6$, $e = f, g$, see Appendix C.

5 Expansions for standardized estimates

Theorem 5.1 gives tools for calculating $f_r(x)$ and $g_r(x)$ of (1.10)-(1.11).

Theorem 5.1 *Expand $[\pi]$, C_{rk} , $e_r(x, L)$ in the form*

$$[\pi] = \sum_{i=0}^{\infty} [\pi]_i n^{-i}, \quad C_{rk} = \sum_{i=0}^{\infty} C_{rki} n^{-i}, \quad e_r(x, L) = \sum_{i=0}^{\infty} e_{ri}(x) n^{-i}.$$

Then, in terms of \overline{A}_{ri} of (1.29), l_{0r} of (1.15), and $L_{0r} = l_{0r}/r!$, $[\pi](L) = [\pi]$, $C_{rk}(L) = C_{rk}$, we have for $e = h, f, g$,

$$\begin{aligned} [\pi]_0 &= [\pi](L_0), \quad C_{rk0} = C_{rk}(L_0), \\ [r] &= L_r = l_r/r!, \\ [r]_i &= \overline{A}_{r,r+i-\delta}, \quad \delta = I(r \geq 3), \\ e_{1i} &= [1]_i + [3]_i H_2 = A_{1,1+i} + \overline{A}_{3,2+i} H_2, \\ h_{ri} &= \sum_{k=r,r+2,\dots,3r} C_{rki} H_{k-1}. \end{aligned}$$

To find h_{r0} , note that when $l_1 = l_2 = 0$, $C_{jj} = 0$ for $j > 0$, so that by (1.33), $C_{r,r+2i} = b_{ri}(\overline{L})$, giving

$$h_{r0} = h_r(x, L_0),$$

where

$$h_r(x, L) = \sum_{i=1}^r b_{ri}(\overline{L}) H_{r+2i-1}.$$

Also,

$$\begin{aligned} \Delta_{1e} &= \Delta_{2e} = 0, \quad \Delta_{3e} = h_{11} = A_{12} + A_{33} H_2/6, \\ \Delta_{4e} &= e_{21}, \quad \Delta_{5e} = e_{12} + e_{31}, \quad \Delta_{6e} = e_{22} + e_{41}. \end{aligned} \tag{5.1}$$

For example, for $i = 1, 2$ and $r \geq 3$,

$$[ir]_1 = \overline{A}_{ii} \overline{A}_{rr} + \overline{A}_{i,i+1} \overline{A}_{r,r-1}$$

and

$$\begin{aligned} [23^2]_1 &= \overline{A}_{22} \overline{A}_{32} \overline{A}_{33} + \overline{A}_{23} \overline{A}_{32}^2/2, \\ [2^2]_1 &= \overline{A}_{22} \overline{A}_{23}, \\ [1^2 4]_1 &= A_{11}^2 \overline{A}_{44}/2! + A_{11} A_{12} \overline{A}_{44}, \quad [1^2 2]_1 = A_{11}^2 \overline{A}_{23}/2 + A_{11} A_{12} \overline{A}_{22}, \\ [123]_1 &= A_{11} \overline{A}_{22} \overline{A}_{33} + A_{11} \overline{A}_{23} \overline{A}_{32} + A_{12} \overline{A}_{22} \overline{A}_{32}, \\ [1^2 3^2]_1 &= A_{11}^2 \overline{A}_{32} \overline{A}_{33}/2 + A_{11} A_{12} \overline{A}_{32}^2/2, \quad [1^2 2]_1 = A_{11}^2 \overline{A}_{23}/2 + A_{11} A_{12} \overline{A}_{22}, \\ [1^3 3]_1 &= A_{11}^3 \overline{A}_{33}/3! + A_{11}^2 A_{12} \overline{A}_{32}/2. \end{aligned}$$

The terms needed for Δ_{rh} for $r = 4, 5, 6$ are

$$\begin{aligned}
r = 4 : \quad & h_{21} = \sum_{k=2,4,6} C_{2k1} H_{k-1}, \\
& C_{221} = [1^2]_1 + \bar{A}_{23}, \quad [1^2]_1 = A_{11} A_{12}, \\
& C_{241} = [4]_1 + [13]_1, \quad [4]_1 = \bar{A}_{44}, \quad [13]_1 = A_{11} \bar{A}_{33} + A_{12} \bar{A}_{32}, \\
& C_{261} = [3^2]_1 = \bar{A}_{32} \bar{A}_{33}. \\
r = 5 : \quad & e_{12} = A_{13} + \bar{A}_{34} H_2 \text{ for } e = h, f, g, \\
& h_{31} = \sum_{k=3,5,7,9} C_{3k1} H_{k-1}, \\
& C_{331} = [1^3]_1 + [12]_1, \quad [1^3]_1 = A_{11}^2 \bar{A}_{12}, \quad [12]_1 = A_{11} \bar{A}_{23} + A_{12} \bar{A}_{22}, \\
& C_{351} = [5]_1 + [14]_1 + [1^2 3]_1 + [23]_1, \quad [5]_1 = \bar{A}_{55}, \\
& [14]_1 = A_{11} \bar{A}_{44} + A_{12} \bar{A}_{43}, \\
& [1^2 3]_1 = A_{11}^2 \bar{A}_{33}/2 + A_{11} A_{12} \bar{A}_{32}, \quad [23]_1 = \bar{A}_{22} \bar{A}_{33} + \bar{A}_{23} \bar{A}_{32}, \\
& C_{371} = [13^2]_1 + [34]_1, \quad [13^2]_1 = A_{11} \bar{A}_{32} \bar{A}_{33} + A_{12} \bar{A}_{32}^2/2, \quad [34]_1 = \bar{A}_{32} \bar{A}_{44} + \bar{A}_{33} \bar{A}_{43}, \\
& C_{391} = [3^3]_1 = \bar{A}_{32}^2 \bar{A}_{33}/2. \\
r = 6 : \quad & h_{22} = \sum_{k=2,4,6} C_{2k2} H_{k-1}, \\
& C_{222} = [1^2]_2 + [2]_2, \quad [1^2]_2 = A_{12}^2/2 + A_{11} A_{13}, \quad [1^2]_2 = \bar{A}_{24}, \\
& C_{242} = [4]_2 + [13]_2, \quad [4]_2 = \bar{A}_{45}, \quad [13]_2 = A_{11} \bar{A}_{34} + A_{12} \bar{A}_{33} + A_{13} \bar{A}_{32}, \\
& C_{262} = [3^3]_2 = \bar{A}_{32} \bar{A}_{34} + \bar{A}_{33}^2/2, \\
& h_{41} = \sum_{k=4,6,8,10,12} C_{4k1} H_{k-1}, \\
& C_{441} = [1^4]_1 + [1^2 2]_1 + [2^2]_1, \quad [1^4]_1 = A_{11}^3 A_{12}/3!, \\
& [1^2 2]_1 = A_{11} A_{12} \bar{A}_{22} + A_{11}^2 \bar{A}_{23}/2, \quad [2^2]_1 = \bar{A}_{22} \bar{A}_{23}, \\
& C_{461} = [6]_1 + [15]_1 + [C_{22} L_4]_1 + [C_{33} L_3]_1, \quad [6]_1 = \bar{A}_{66}, \quad [15]_1 = A_{11} \bar{A}_{55} + A_{12} \bar{A}_{54}, \\
& [C_{22} L_4]_1 = A_{43} C_{221} + [4]_1 C_{220}, \quad [4]_1 = \bar{A}_{44}, \\
& [C_{33} L_3]_1 = \bar{A}_{32} C_{331} + \bar{A}_{33} C_{330}, \\
& C_{220} = A_{11}^2/2! + \bar{A}_{22}, \quad C_{330} = A_{11}^3/3! + A_{11} \bar{A}_{22}, \\
& C_{481} = C_{220} [3^2]_1 + C_{221} [3^2]_0 + [134]_1 + [4^2]_1 + [35]_1, \\
& [3^2]_0 = \bar{A}_{32}^2/2!, \quad [134]_1 = A_{11} \bar{A}_{32} \bar{A}_{44} + A_{11} \bar{A}_{33} \bar{A}_{43} + A_{12} \bar{A}_{32} \bar{A}_{43}, \\
& [4^2]_1 = \bar{A}_{43} \bar{A}_{44}, \quad [35]_1 = \bar{A}_{32} \bar{A}_{55} + \bar{A}_{33} \bar{A}_{54}, \\
& C_{4,10,1} = [13^3]_1 + [3^2 4]_1, \quad [13^3]_1 = 3 A_{11} \bar{A}_{32}^2 \bar{A}_{33} + A_{12} \bar{A}_{32}^3/3!, \\
& [3^2 4]_1 = \bar{A}_{32}^2 \bar{A}_{44}/2! + \bar{A}_{33} \bar{A}_{32} \bar{A}_{43}, \\
& C_{4,12,1} = [3^4]_1 = \bar{A}_{32}^3 \bar{A}_{33}/3!.
\end{aligned} \tag{5.2}$$

By (5.1), the terms needed for Δ_{re} for $e = f, g$ and $r = 4, 5, 6$ are given by (5.2) for $r = 5$,

and

$$\begin{aligned}
r = 4 : e_{21} &= \overline{A}_{23}H_1 + [4]_1 H_3 + [3^2]_1 e(3^2) + [13]_1 e(13), \\
r = 5 : e_{31} &= \sum \{[\pi]_1 e(\pi) : \pi \in S_3(e)\}, \\
r = 6 : e_{22} &= \sum \{[\pi]_2 e(\pi) : \pi \in S_2(e)\}, \\
e_{41} &= \sum \{[\pi]_1 e(\pi) : \pi \in S_4(e)\},
\end{aligned}$$

where $S_r(e)$ is given in Appendix C for $e = f, g$.

Using Theorem 5.1, g_{12} is given by (5.2), and

$$\Delta_{5g} = g_{12} + g_{31},$$

where

$$g_{31} = \sum \{[\pi]_1 g(\pi), \pi \in S_3(g)\}, S_3(g) = \{5, 34, 3^3, 23\}.$$

By Table 1.1, g_4 has 8+3 terms, or 2+2 terms for X_{12} and 2+1 terms for X_{23} . For $r \leq 3$, Δ_{re} does not depend on e . If $l_{03} = 0$, then

$$\Delta_{2h} - l_{01}A_{12} = \Delta_{2f} = \Delta_{2g} = A_{23}H_1/2 + A_{45}H_3/4!.$$

For $e = f, g$, e_{22} needs $[4]_2 = \overline{A}_{45}$, $[2]_2 = \overline{A}_{24}$, $[3^2]_2$ above and

$$[13]_2 = A_{11}\overline{A}_{34} + A_{12}\overline{A}_{33} + A_{13}\overline{A}_{32}.$$

For g_{41} needs $[\pi]_1$ for $\pi \in S_4(g) = \{6, 4^2, 24, 2^2, 35, 3^24, 3^4, 23^2\}$: f_{41} needs $[\pi]_1$ for $\pi \in S_4(f) = S_4(g) \cup \{15, 1^24, 1^22, 134, 13^3, 123, 1^23^2, 1^33\}$.

6 Cumulant coefficients for $Y_{JK\theta}$ of (1.18)

Theorem 6.1 gives $A_{ri\theta}^{JK}$ of (1.19) in terms of $A_{ri\theta}$ of (1.4). Its proof is outlined as follows: Write $s = s_{2K\theta}$ of (1.17) as $a_{21\theta}n^{-1}(1 + \epsilon)$, where $\epsilon = \sum_{j=1}^{K-1} x_j n^{-j}$, $x_j = a_{2,j+1,\theta}$. So,

$$(1 + \epsilon)^{-r/2} = \sum_{k=0}^{\infty} \binom{-r/2}{k} \epsilon^k.$$

By (A.1),

$$\epsilon^k = \sum_{j=k}^{\infty} \widehat{B}_{jk}(x) n^{-j}.$$

So,

$$s^{-r/2} = (a_{21\theta}n^{-1})^{-r/2} \sum_{j=0}^{\infty} d_{rj} n^{-j},$$

where

$$d_{rj} = \sum_{k=0}^j \binom{-r/2}{k} \widehat{B}_{jk}(x)$$

depends on

$$\begin{aligned} d_{r0} &= 1, d_{r1} = (-r/2)x_1, d_{r2} = (-r/2)x_2 + \binom{-r/2}{2}x_1^2, \\ x_1 &= A_{22\theta} I(1 < K), x_2 = A_{23\theta} I(2 < K), \dots \end{aligned}$$

Theorem 6.1 and Corollary 6.1 are now immediate.

Theorem 6.1 *With notation as above,*

$$\begin{aligned} A_{1i\theta}^{JK} &= 0 \text{ for } i \leq J, \\ A_{1i\theta}^{JK} &= A_{1i\theta} \otimes d_{1i} = \sum_{j=J+1}^i d_{1,i-j} A_{1j\theta}, \quad i \geq J+1, \\ A_{ri\theta}^{JK} &= A_{ri\theta} \otimes d_{ri} = \sum_{j=r-1}^i d_{r,i-j} A_{rj\theta}, \quad r \geq \max(2, i-1), \end{aligned} \tag{6.1}$$

where $a_i \otimes b_i = \sum_{j=0}^i a_j b_{i-j}$.

Corollary 6.1 *With notation as above,*

$$A_{1,J+1,\theta}^{JK} = A_{1,J+1,\theta}, \quad A_{1,J+2,\theta}^{JK} = A_{1,J+1,\theta} d_{11} + A_{1,J+2,\theta}$$

and for $r \geq 2$,

$$\begin{aligned} A_{r,r-1,\theta}^{JK} &= A_{r,r-1,\theta}, \quad A_{rr\theta}^{JK} = A_{rr\theta} + d_{r1} A_{r,r-1,\theta}, \\ A_{r,r+1,\theta}^{JK} &= A_{r,r+1,\theta} + d_{r1} A_{rr\theta} + d_{r2} A_{r,r-1,\theta}. \end{aligned}$$

For $r = 2$ it is simpler to use

$$\begin{aligned} A_{2i\theta}^{JK} &= 0 \text{ for } 2 \leq i \leq K, \\ &= \sum_{j=K+1}^i d_{2,i-j} A_{2j\theta} \text{ for } i > K. \end{aligned}$$

The last result of the corollary follows since $\kappa_2(Y_{JK\theta}) - 1 = s_{2\infty\theta}/s_{2K\theta} - 1 = s_{2K\theta}^{-1} \sum_{j=K+1}^{\infty} a_{2j\theta} n^{-j}$.

7 Expansions for the density of $Y_{01\theta}$ and its derivatives

Withers and Nadarajah (2012) gave expansions for the density and other derivatives of the distribution. Theorem 7.1 gives expansions under the extended Cornish and Fisher assumption, $l_r = O(1)$ for $r \geq 1$, and under the more general assumption (1.3).

Theorem 7.1 *Under the extended Cornish and Fisher assumption, $l_r = O(1)$ for $r \geq 1$, expansions for the density of $Y_{01\theta}$, p_n , and its derivatives follow from (1.5) and (3.2):*

$$(-D)^i p_n(x) = p(x) \sum_{r=1}^{\infty} n^{-r/2} h_{ir}(x, L), \quad i \geq 0,$$

where

$$h_{ir}(x, L) = \sum_{k=1}^{3r} C_{rk} H_{k+i}(x),$$

where h_{ri} has a different meaning from that in (1.12)-(1.14). Under the more general assumption (1.3), the expansions take the form

$$(-D)^i p_n(x) = p(x) \sum_{r=1}^{\infty} n^{-r/2} h_{ir}(x), \quad i \geq 0,$$

where $h_{ir}(x)$ is $h_r(x)$ with $\{H_k\}$ replaced by $\{H_{k+i+1}\}$. For example,

$$h_{ir}(x) = h_{ir}(x, l) + \Delta_{rih},$$

where

$$\begin{aligned} \Delta_{rih} &= 0 \text{ for } r = 1, 2, \\ \Delta_{3ih} &= A_{12}H_{i+1} + \bar{A}_{33}H_{i+3}, \\ \Delta_{4ih} &= (A_{11}A_{12} + \bar{A}_{23})H_{i+2} + (A_{12}\bar{A}_{32} + A_{11}\bar{A}_{33} + \bar{A}_{44})H_{i+4} + \bar{A}_{32}\bar{A}_{33}H_{i+6}. \end{aligned}$$

This follows from its Charlier expansion version given in equation (2.6) of Withers and Nadarajah (2012), so it remains valid for general X . See equation (4.6) there for the case $X = N$. For its multivariate extension, see Section 7 of Withers and Nadarajah (2011).

Appendix A: The ordinary Bell polynomials

The *ordinary Bell polynomial*, $\hat{B}_{rj}(y)$, is defined in terms of a sequence $y = (y_1, y_2, \dots)$, by

$$S(t)^j = \sum_{r=j}^{\infty} \hat{B}_{rj}(y) t^r \quad (\text{A.1})$$

for $j = 0, 1, \dots$, where

$$S(t) = \sum_{r=1}^{\infty} y_r t^r.$$

In fact, for $r > 0$, $\hat{B}_{rj}(y)$ is only a function of $\{x_i, 1 \leq i \leq r - j + 1\}$. For example,

$$\begin{aligned} \hat{B}_{rj}(y) &= 0 \text{ for } r < j, \quad \hat{B}_{r0}(y) = I(r = 0), \\ \hat{B}_{rj}(-y) &= (-1)^j \hat{B}_{rj}(y), \\ \hat{B}_{r1}(y) &= y_r, \quad \hat{B}_{rr}(y) = y_1^r, \quad \hat{B}_{r+1,r}(y) = r y_1^{r-1} y_2, \\ \hat{B}_{r+2,r}(y) &= r y_1^{r-1} y_3 + \binom{r}{2} y_1^{r-2} y_2^2. \end{aligned}$$

They are tabled on page 309 of Comtet (1974) for $1 \leq j \leq r \leq 10$. Setting

$$[1^{i_1} 2^{i_2} \dots] = \left(y_1^{i_1} / i_1! \right) \left(y_2^{i_2} / i_2! \right) \dots,$$

we can write \widehat{B}_{rj} and b_{rj} of (1.33) as

$$b_{rj}(y) = \widehat{B}_{rj}(y) / j! = \sum [1^{i_1} 2^{i_2} \dots] \quad (\text{A.2})$$

summed over all partitions of r , that is, over $1i_1 + 2i_2 + \dots = r$, subject to $i_1 + i_2 + \dots = j$.

Taking the coefficient of t^r in $S(t)^{j+k} = S(t)^j S(t)^k$ gives the recurrence formula

$$\widehat{B}_{r,j+k}(y) = \widehat{B}_{rj}(y) \otimes \widehat{B}_{rk}(y) = \sum_{a+b=r} \widehat{B}_{aj}(y) \widehat{B}_{bk}(y).$$

For example, $k = 1$ gives

$$\widehat{B}_{r,j+1}(y) = \sum_{a=j}^{r-1} \widehat{B}_{aj}(y) y_{r-a}, \quad r \geq j+1,$$

and $j = k = 1$ gives

$$\begin{aligned} b_{r2}(y) = \widehat{B}_{r2}(y/2) &= \sum_{a=1}^{r-1} y_a y_{r-a} / 2, \quad r \geq 2, \\ &= \begin{cases} \sum_{a=1}^b y_a y_{r-a}, & \text{if } r = 2b+1, \\ x_b^2 / 2 + \sum_{a=1}^{b-1} y_a y_{r-a}, & \text{if } r = 2b. \end{cases} \end{aligned}$$

The *exponential Bell polynomial*, $B_{rj}(x)$, is the coefficient of $t^r / r!$ in $S(t)^j / j!$ when $y_j = x_j / j!$. So,

$$B_{rj}(x) / r! = \widehat{B}_{rj}(y) / j! = b_{rj}(y).$$

They are tabled on pages 307, 308 of Comtet (1974) for $1 \leq j \leq r \leq 12$.

It is easy to show that if $Y_r = ab^r y_r$, $X_r = ab^r x_r$ then

$$\widehat{B}_{rj}(Y) = a^j b^r \widehat{B}_{rj}(y), \quad B_{rj}(X) = a^j b^r B_{rj}(x). \quad (\text{A.3})$$

The multinomial expansion can be written as $S(1)^r / r! = \sum [1^{i_1} 2^{i_2} \dots]$ summed over all partitions of r .

Appendix B: Derivatives of H_r , and H_r as a function of a

We first show that H_r has k th derivative

$$H_{r \cdot k} = \sum_{i=0}^k \binom{k}{i} (-1)^i b_{k-i} H_{r+i}, \quad k \geq 0, \quad (\text{B.1})$$

where

$$b_i = (H_1 + D)^i 1 = (H_1 + D) b_{i-1}.$$

In particular,

$$\begin{aligned} b_0 &= 1, \quad b_1 = H_1, \quad b_2 = 2H_1^2 - H_2, \quad b_3 = 3!H_1^3 - 6H_1H_2 + H_3, \\ b_4 &= 4!H_1^4 - 36H_1^2H_2 + 8H_1H_3 + 6H_2^2 - H_4, \\ b_5 &= 5!H_1^5 - 240H_1^3H_2 + 60H_1^2H_3 + 90H_1H_2^2 - 10H_1H_4 - 20H_2H_3 + H_5. \end{aligned}$$

We also prove (1.35), giving b_k in terms of \mathbf{a} .

Rewriting (1.37) as

$$H_{r+1} = H_1 H_r - H_{r+1}, \quad r \geq 0,$$

we obtain

$$H_{r+k} = \sum_{i=0}^k \alpha_{ki} H_{r+i}, \quad k \geq 0,$$

where

$$\alpha_{k+1,i} = (H_1 + D) \alpha_{ki} - \alpha_{k,i-1},$$

where $\alpha_{ki} = 0$ if $i < 0$. The second equation follows from $H_{r+k+1} = DH_{r+k}$. This gives

$$\begin{aligned} \alpha_{kk} &= (-1)^k, \quad \alpha_{k,k-1} = (-1)^{k-1} k H_1, \\ \alpha_{k,k-i} &= (-1)^{k-i} \binom{k}{i} b_i, \quad b_i = \alpha_{i0}. \end{aligned}$$

But by (1.37),

$$\begin{aligned} H_r &= (a_1 - D) H_{r-1} = (a_1 - D)^r 1, \quad r \geq 1, \\ &= (-1)^r B_r(-\mathbf{a}) = \sum_{i=0}^r (-1)^{r-i} B_{ri}(\mathbf{a}), \quad r \geq 0. \end{aligned} \tag{B.2}$$

In particular,

$$\begin{aligned} H_1 &= a_1, \quad H_2 = a_1^2 - a_2, \quad H_3 = a_1^3 - 3a_1a_2 + a_3, \\ H_4 &= a_1^4 - 6a_1^2a_2 + 3a_2^2 + 4a_1a_3 - a_4, \end{aligned} \tag{B.3}$$

$$H_5 = a_1^5 - 10a_1^3a_2 + 15a_1a_2^2 + 10a_1^2a_3 - 10a_2a_3 - 5a_1a_4 + a_5, \tag{B.4}$$

$$\begin{aligned} H_6 &= a_1^6 - 15a_1^4a_2 + 45a_1^2a_2^2 - 15a_2^3 + 20a_1^3a_3 - 60a_1a_2a_3 + 10a_2^2a_3 \\ &\quad - 15a_1^2a_4 + 15a_2a_4 + 6a_1a_5 - a_6 - 10a_2a_3 + 10a_1^2a_3 - 10a_1^3a_2 + a_1^5. \end{aligned} \tag{B.5}$$

So, Comtet (1974)'s table gives H_r in terms of \mathbf{a} to $r = 12$. Replacing \mathbf{a} by $-\mathbf{a}$ gives

$$b_r = B_r(\mathbf{a}) = \sum_{i=0}^r B_{ri}(\mathbf{a}), \quad r \geq 0,$$

proving (1.35). $B_r(\mathbf{a})$ is called *the r th complete exponential Bell polynomial*. So, Comtet (1974)'s table pages 307-308 gives b_r in terms of \mathbf{a} up to $r = 12$. The first six are

$$\begin{aligned} b_0 &= 1, \quad b_{10} = a_1, \quad b_2 = a_2 + a_1^2, \quad b_3 = a_3 + 3a_1a_2 + a_1^3, \\ b_4 &= a_4 + 4a_1a_3 + 3a_2^2 + 6a_1^2a_2 + a_1^4, \\ b_5 &= a_5 + 5a_1a_4 + 10a_2a_3 + 10a_1^2a_3 + 15a_1a_2^2 + 10a_1^3a_2 + a_1^5, \\ b_6 &= a_6 + 6a_1a_5 + 15a_2a_4 + 10a_3^2 + 15a_1^2a_4 + 60a_1a_2a_3 + 15a_2^3 \\ &\quad + 20a_1^3a_3 + 45a_1^2a_2^2 + 15a_1^4a_2 + a_1^6. \end{aligned}$$

For example, this gives the first four derivatives of H_r in terms of α_{44} , α_{43} above and $\alpha_{42} = 6b_2$, $\alpha_{41} = -4b_3$.

Expressions for a_r in terms of \mathbf{H} : By (B.1),

$$a_r = H_{1..r-1} = \sum_{i=0}^{r-1} \binom{r-1}{i} (-1)^i b_{r-1-i} H_{1+i}, \quad r \geq 1.$$

In particular,

$$\begin{aligned} a_1 &= H_1, \quad a_2 = H_1^2 - H_2, \quad a_3 = 2H_1^3 - 3H_1H_2 + H_3, \\ a_4 &= 3!H_1^4 - 12H_1^2H_2 + 4H_1H_3 + 3H_2^2 - H_4, \\ a_5 &= 4!H_1^5 - 60H_1^3H_2 + 20H_1^2H_3 + 30H_1H_2^2 - 5H_1H_4 - 10H_2H_3 + H_5, \\ a_6 &= 5!H_1^6 - 360H_1^4H_2 + 120H_1^3H_3 - 30H_1^2H_4 + 6H_1H_5 - H_6 + 270H_1^2H_2^2 \\ &\quad - 120H_1H_2H_3 - 30H_2^3 + 15H_2H_4. \end{aligned}$$

A simpler way to express a_r in terms of $\{H_i\}$ is to apply Faa di Bruno's rule, [4i] of Comtet (1974), to obtain the r th derivative of $a(x) = f(p(x))$ at $f(p) = -\ln p$, in terms of $\mathbf{p} = (p_1, p_2, \dots)$, where $p_r = D^r p(x)$:

$$a_r = \sum_{j=1}^r f_j B_{rj}(\mathbf{p}), \quad r \geq 1,$$

where $f_j = F_j(-p)^{-j}$ at $F_j = (j-1)!$, $p = p(x)$ is the j th derivative of $f(p)$ at p . But $p_r = ab^r H_r$, where $b = -1$, $a = p$. So, by (A.3), $B_{rj}(\mathbf{p}) = p^j (-1)^r B_{rj}(\mathbf{H})$. This gives the simple inverse formula

$$a_r = \sum_{j=1}^r (-1)^{r-j} (j-1)! B_{rj}(\mathbf{H}), \quad r \geq 1.$$

So, Comtet (1974)'s table pages 307-308 gives a_r in terms of \mathbf{H} up to $r = 12$. For example,

$$a_6 = -B_{61} + B_{62} - 2B_{63} + 3!B_{64} - 4!B_{65} + 5!B_{66},$$

where

$$\begin{aligned} B_{61} &= H_6, \quad B_{62} = 6H_1H_5 + 15H_2H_4, \quad B_{63} = 15H_1^2H_4 + 60H_1H_2H_3 + 15H_2^3, \\ B_{64} &= 20H_1^3H_3 + 45H_1^2H_2^2, \quad B_{65} = 15H_1^4H_2, \quad B_{66} = H_1^6. \end{aligned}$$

This is the inverse formula to (B.2). These two relations are essentially the relations between the non-central moments and the cumulants: $-a = \ln p$, $p = e^{-a}$, $K = \ln M$, $M = e^K$, where M , K are the moment and cumulant generating functions. So, $-a_r$ can be identified with the r th cumulant, and H_r with the r th moment.

An alternative way to express b_r in terms of $\{a_i\}$ of (1.26), is to set

$$e_1 = 0, \quad e_r = \sum_{i=2}^r \binom{r}{i} a_1^{r-i} a_i \text{ if } r \geq 2, \quad \delta_r = b_r - a_1^r - e_r.$$

Then,

$$\delta_2 = \delta_3 = 0, \quad \delta_{r+1} = (a_1 + D) \delta_r + r a_2 e_{r-1}, \quad r \geq 1.$$

This gives

$$\begin{aligned} b_r &= a_1^r + e_r + \delta_r, \quad r \leq 6, \\ \delta_4 &= 3a_2^2, \\ \delta_5 &= 15a_1 a_2^2 + 10a_2 a_3, \\ \delta_6 &= 45a_1^2 a_2^2 + 60a_1 a_2 a_3 + 15a_2^3 + 15a_2 a_4 + 10a_3^2. \end{aligned}$$

Appendix C: $e_r(x, L)$ for $e = f, g$ in terms of \mathbf{H}

(1.34) gives $f_r(x, L)$, $g_r(x, L)$, $r \leq 6$ in terms of certain $e(\pi)$. We now give these in terms of \mathbf{H} . First consider the case $l_3 = 0$. Let $S_{0r}(e)$ denote the partitions in $S_r(e)$ needed when $l_3 = 0$. For each r we first give $e(\pi)$ covered by the special cases (4.4), (4.5), (4.6).

$$\begin{aligned} r = 2 : \quad & S_{02}(f) = \{4, 2\}, \quad f(4) = H_3, \quad f(2) = H_1. \\ r = 3 : \quad & S_{03}(f) = \{5, 14, 12\}, \quad f(5) = H_4, \quad f(12) = H_2 - H_1^2 = -H_{1,1} \text{ of (4.5), (4.6),} \\ & f(14) = H_4 - H_1 H_3 = -H_{3,1}. \\ r = 4 : \quad & S_{04}(f) = \{6, 4^2, 24, 2^2, 15, 1^2 4, 1^2 2\}, \quad f(6) = H_5, \quad f(1^2 2) \text{ of (4.6),} \\ & f(4^2) = H_7 - H_1 H_3^2, \\ & f(24) = H_5 - H_1^2 H_3, \quad f(2^2) = H_3 - H_1^3, \quad f(15) = H_5 - H_1 H_4 = -H_{4,1} \text{ of (4.5),} \\ & f(1^2 4) = H_5 - H_2 H_3 - 2H_1 H_4 + 2H_1^2 H_3. \\ r = 5 : \quad & S_{05}(f) = \{7, 45, 25, 16, 14^2, 124, 12^2, 1^2 5, 1^3 4, 1^3 2\}, \quad f(7) = H_6, \quad f(1^3 2) \text{ of (4.6),} \\ & f(45) = H_8 - H_1 H_3 H_4, \quad f(25) = H_6 - H_1^2 H_4, \quad f(16) = H_6 - H_1 H_5 = -H_{5,1} \text{ of (4.5),} \\ & f(14^2) = H_8 - H_1 H_7 - 2H_1 H_3 H_4 - H_2 H_3^2 + 3H_1^2 H_3^2, \\ & f(124) = H_6 - H_1 H_5 - 2H_1 H_2 H_3 - H_1^2 H_4 + 3H_1^3 H_3, \\ & f(12^2) = H_4 - H_1 H_3 - 3H_1^2 H_2 + 3H_1^4, \\ & f(1^2 5) = H_6 - H_2 H_4 - 2H_1 H_5 + 2H_1^2 H_4, \\ & f(1^3 4) = H_6 - 3H_1 H_5 - 3H_2 H_4 + 6H_1^2 H_4 - H_3^2 + 6H_1 H_2 H_3 - 6H_1^3 H_3. \end{aligned}$$

For $r = 6$, because of the increasing number of terms, we adapt the notation of Comtet (1974), setting

$$k \cdot 1^{i_1} 2^{i_2} \dots \stackrel{H}{=} k H_1^{i_1} H_2^{i_2} \dots \quad (\text{C.1})$$

when giving formulas for $f(\pi)$, $g(\pi)$. For example,

$$5 \cdot 1^4 6 \stackrel{H}{=} 5H_1^4 H_6, \quad 15 \cdot 1^3 2^2 \stackrel{H}{=} 15H_1^3 H_2^2, \quad 3 \cdot 4(11) \stackrel{H}{=} 3H_4 H_{11}.$$

Using this notation, when $l_3 = 0$, $f_6(x, L)$ is given by

$r = 6 :$

$$S_{06}(f) = \{8, 5^2, 46, 4^3, 26, 24^2, 2^2 4, 2^3, 17, 145, 125, 1^2 6, 1^2 4^2, 1^2 2^2, 1^3 5, 1^4 4, 1^4 2, 1^6\},$$

$$f(8) = H_7 \stackrel{H}{=} 7, \quad f(1^4 2) \text{ of (4.6)}, \quad f(5^2) \stackrel{H}{=} 9 - 14^2, \quad f(46) \stackrel{H}{=} 9 - 135,$$

$$f(4^3) \stackrel{H}{=} 11 - 3 \cdot 137 - 23^3 + 3 \cdot 1^2 3^3,$$

$$f(26) \stackrel{H}{=} 7 - 1^2 5,$$

$$f(24^2) \stackrel{H}{=} 9 - 1^2 7 - 2 \cdot 135 - 123^2 + 3 \cdot 1^3 3^2,$$

$$f(2^2 4) \stackrel{H}{=} 7 - 2 \cdot 1^2 5 - 13^2 - 1^2 23 + 3 \cdot 1^4 3,$$

$$f(2^3) \stackrel{H}{=} 5 - 3 \cdot 1^2 3 - 1^3 2 + 3 \cdot 1^5,$$

$$f(17) \stackrel{H}{=} 7 - 16 = -H_{6.1} \text{ of (4.5)},$$

$$f(145) \stackrel{H}{=} 9 - 18 - 135 - 14^2 - 234 + 3 \cdot 1^2 34,$$

$$f(125) \stackrel{H}{=} 7 - 16 - 1^2 5 - 2 \cdot 124 + 3 \cdot 1^3 4,$$

$$f(1^2 6) \stackrel{H}{=} 7 - 2 \cdot 16 - 25 + 2 \cdot 1^2 5,$$

$$f(1^2 4^2) \stackrel{H}{=} 9 - 2 \cdot 18 - 27 + 2 \cdot 1^2 7 - 2 \cdot 135 - 2 \cdot 14^2 - 4 \cdot 234 + 12 \cdot 1^2 34 \\ - 3^3 + 9 \cdot 123^2 - 12 \cdot 1^3 3^2,$$

$$f(1^2 24) \stackrel{H}{=} 7 - 2 \cdot 16 - 25 + 1^2 5 - 4 \cdot 124 + 6 \cdot 1^3 4 - 2 \cdot 13^2 - 2 \cdot 2^2 3 + 15 \cdot 1^2 23 - 12 \cdot 1^4 3,$$

$$f(1^2 2^2) \stackrel{H}{=} 5 - 2 \cdot 14 - 23 - 1^2 3 - 6 \cdot 12^2 + 21 \cdot 1^3 2 - 12 \cdot 1^5,$$

$$f(1^3 5) \stackrel{H}{=} 7 - 3 \cdot 16 - 3 \cdot 25 + 6 \cdot 1^2 5 - 34 + 6 \cdot 124 - 6 \cdot 1^3 4,$$

$$f(1^4 4) \stackrel{H}{=} 7 - 4 \cdot 16 - 6 \cdot 25 + 12 \cdot 1^2 5 - 5 \cdot 34 + 24 \cdot 124 - 24 \cdot 1^3 4 + 8 \cdot 13^2 + 6 \cdot 2^2 3 \\ - 36 \cdot 1^2 23 + 24 \cdot 1^4 3.$$

A second special case for $g(\pi)$ is

$$g(2k) = H_{k+1} - H_1 H_k - H_{k-1} (H_2 - H_1^2), \quad k \geq 1. \quad (\text{C.2})$$

For example, $g(24) = H_5 - H_1H_4 - H_2H_3 + H_1^2H_3$. When $l_3 = 0$, $g_r(x, L)$ is given by (1.34) as follows:

$$\begin{aligned}
r = 2 : & \quad S_{02}(g) = \{4, 2\}, \quad g(4) = H_3, \quad g(2) = H_1. \\
r = 3 : & \quad S_{03}(g) = \{5\}, \quad g(5) = H_4. \\
r = 4 : & \quad S_{04}(g) = \{6, 4^2, 24, 2^2\}, \quad g(6) = H_5, \quad g(24), g(2^2) \text{ of (C.2)}, \\
& \quad g(4^2) = H_7 - 2H_3H_4 + H_1H_3^2. \\
r = 5 : & \quad S_{05}(g) = \{25, 45, 7\}, \quad g(7) = H_6, \quad g(25) \text{ of (C.2)}, \\
& \quad g(45) = H_8 - H_3H_5 - H_4^2 + H_1H_3H_4. \\
r = 6 : & \quad S_{06}(g) = \{8, 5^2, 46, 4^3, 26, 24^2, 2^24, 2^3\}, \quad g(8) = H_7 \stackrel{H}{=} 7, \quad g(26) \text{ of (C.2)}, \\
& \quad g(5^2) \stackrel{H}{=} 9 - 2 \cdot 45 + 14^2, \\
& \quad g(46) \stackrel{H}{=} 9 - 36 - 45 + 135, \\
& \quad g(4^3) \stackrel{H}{=} 11 - 3 \cdot 38 - 3 \cdot 47 + 3 \cdot 137 + 3 \cdot 3^25 + 6 \cdot 34^2 - 9 \cdot 13^24 - 23^3 + 3 \cdot 1^23^3, \\
& \quad g(24^2) \stackrel{H}{=} 9 - 18 - 27 + 1^27 - 2 \cdot 36 - 2 \cdot 45 + 4 \cdot 135 + 2 \cdot 14^2 + 4 \cdot 234 - 6 \cdot 1^234 \\
& \quad + 3^3 - 4 \cdot 123^2 + 3 \cdot 1^33^2, \\
& \quad g(2^24) \stackrel{H}{=} 7 - 2 \cdot 16 - 2 \cdot 25 + 3 \cdot 1^25 - 2 \cdot 34 + 4 \cdot 124 - 3 \cdot 1^34 \\
& \quad + 3 \cdot 13^2 + 2 \cdot 2^23 - 7 \cdot 1^223 + 3 \cdot 1^43, \\
& \quad g(2^3) \stackrel{H}{=} 5 - 3 \cdot 14 - 3 \cdot 23 + 6 \cdot 1^23 + 6 \cdot 12^2 - 10 \cdot 1^32 + 3 \cdot 1^5.
\end{aligned}$$

When $l_3 = 0$,

$$h_2(x, L) - [1^2] H_1 = f_2(x, L) = g_2(x, L) = [2]H_1 + [4]H_3.$$

For the case $X = N$, it is known that $g_r(x, L) - I(r = 1)L_1$ does not depend on L_1 . This is a key step used in of Withers (1989a, 1983) to construct parametric and non-parametric confidence intervals of level say $0.95 + O(n^{-r/2})$, given $r \geq 1$. We have seen that this property is also true for general X for $r \leq 6$. We now show that it is true for all r . Set

$$s = \overline{P}_n(y) = \Pr(Y_{01\theta} - \lambda_{1n} \leq y) = P_n(y + \lambda_{1n}) = P(x)$$

say. Setting $g_0(x) = x$ gives

$$\sum_{r=0}^{\infty} n^{-r/2} g_r(x) = P_n^{-1}(s) = \lambda_{1n} + y = \lambda_{1n} + \overline{P}_n^{-1}(s) = n^{-1/2}l_1 + \sum_{r=0}^{\infty} n^{-r/2} [g_r(x)]_{l_1=0}.$$

Taking the coefficient of $n^{-r/2}$ gives $g_r(x, L) = I(r = 1)L_1 + [g_r(x)]_{l_1=0}$.

We now give the extra terms needed when $l_3 \neq 0$.

$$\begin{aligned}
r = 2 : & \\
& \quad f(3^2) = H_5 - H_1H_2^2, \quad f(13) = H_3 - H_1H_2 = -H_{2,1} \text{ of (4.5)}. \\
r = 3 : & \\
& \quad f(34) = H_6 - H_1H_2H_3, \\
& \quad f(3^3) = H_8 - 3H_1H_2H_5 - H_2^4 + 3H_1^2H_2^3, \\
& \quad f(23) = H_4 - H_1^2H_2, \\
& \quad f(13^2) = H_6 - H_1H_5 - 2H_1H_2H_3 - H_2^3 + 3H_1^2H_2^2, \\
& \quad f(1^23) = H_4 - 2H_1H_3 - H_2^2 + 2H_1^2H_2.
\end{aligned}$$

$r = 4 :$

$$f(35) \stackrel{H}{=} 7 - 124,$$

$$f(3^2 4) = 9 - 2 \cdot 126 - 135 - 2^3 3 + 3 \cdot 1^2 2^2 3,$$

$$f(3^4) \stackrel{H}{=} 11 - 4 \cdot 128 - 3 \cdot 15^2 - 6 \cdot 2^3 5 + 18 \cdot 1^2 2^2 5 - 2^4 3 + 10 \cdot 12^5 - 15 \cdot 1^3 2^4,$$

$$f(23^2) \stackrel{H}{=} 7 - 2 \cdot 124 - 12^3 - 1^2 5 + 3 \cdot 1^3 2^2,$$

$$f(134) \stackrel{H}{=} 7 - 16 - 124 - 13^2 - 2^2 3 + 3 \cdot 1^2 23,$$

$$f(13^3) \stackrel{H}{=} 9 - 18 - 3 \cdot 126 - 3 \cdot 135 - 3 \cdot 2^2 5 + 9 \cdot 1^2 25 - 4 \cdot 2^3 3 \\ + 9 \cdot 1^2 2^2 3 + 10 \cdot 12^4 - 15 \cdot 1^3 2^3,$$

$$f(123) \stackrel{H}{=} 5 - 14 - 1^2 3 - 2 \cdot 12^2 + 3 \cdot 1^3 2,$$

$$f(1^2 3^2) \stackrel{H}{=} 7 - 2 \cdot 16 - 25 + 2 \cdot 1^2 5 - 2 \cdot 124 - 2 \cdot 13^2 - 5 \cdot 2^2 3 \\ + 9 \cdot 12^3 + 12 \cdot 1^2 23 - 12 \cdot 1^3 2^2,$$

$$f(1^3 3) \stackrel{H}{=} 5 - 3 \cdot 14 - 4 \cdot 23 + 6 \cdot 1^2 3 + 6 \cdot 12^2 - 6 \cdot 1^3 2.$$

$r = 5 :$

$$f(36) \stackrel{H}{=} 8 - 125,$$

$$f(34^2) \stackrel{H}{=} 10 - 2 \cdot 136 - 127 - 2^2 3^2 + 3 \cdot 1^2 23^2,$$

$$f(3^2 5) \stackrel{H}{=} 10 - 2 \cdot 127 - 145 + 3 \cdot 1^2 2^2 4 - 2^3 4,$$

$$f(3^3 4) \stackrel{H}{=} 12 - 3 \cdot 129 - 138 - 3 \cdot 156 + 9 \cdot 1^2 2^2 6 - 3 \cdot 2^3 6 - 2^3 3^2 + 9 \cdot 1^2 235 - 3 \cdot 2^2 35 \\ + 10 \cdot 12^4 3 - 15 \cdot 1^3 2^3 3,$$

$$f(3^5) \stackrel{H}{=} (14) - 5 \cdot 12(11) - 10 \cdot 158 - 10 \cdot 2^3 8 + 30 \cdot 1^2 2^2 8 \\ - 15 \cdot 2^2 5^2 + 45 \cdot 1^2 25^2 - 10 \cdot 2^3 35 \\ + 100 \cdot 12^4 5 - 150 \cdot 1^3 2^3 5 - 2^5 4 + 15 \cdot 12^5 3 + 10 \cdot 2^7 - 105 \cdot 1^2 2^6 + 105 \cdot 1^4 2^5,$$

$$f(234) \stackrel{H}{=} 8 - 1^2 6 - 125 - 134 - 12^2 3 + 3 \cdot 1^3 23,$$

$$f(23^3) \stackrel{H}{=} 10 - 1^2 8 - 3 \cdot 127 - 3 \cdot 145 + 9 \cdot 1^3 25 - 3 \cdot 12^2 5 + 9 \cdot 1^2 2^2 4 \\ - 3 \cdot 2^3 4 - 12^3 3 + 10 \cdot 1^2 2^4 - 15 \cdot 1^4 2^3,$$

$$f(2^2 3) \stackrel{H}{=} 6 - 123 - 2 \cdot 1^2 4 - 1^2 2^2 + 3 \cdot 1^4 2,$$

$$f(135) \stackrel{H}{=} 8 - 17 - 125 - 134 - 2^2 4 + 3 \cdot 1^2 24,$$

$$f(13^2 4) \stackrel{H}{=} (10) - 19 - 2 \cdot 127 - 3 \cdot 136 - 2 \cdot 2^2 6 + 6 \cdot 1^2 26 - 145 \\ + 3 \cdot 1^2 35 - 235 - 2^3 4 + 3 \cdot 1^2 2^2 4 \\ - 3 \cdot 2^2 3^2 + 6 \cdot 1^2 23^2 + 10 \cdot 12^3 3 - 15 \cdot 1^3 2^2 3,$$

$$f(13^4) \stackrel{H}{=} (12) - 1(11) - 4 \cdot 129 - 4 \cdot 138 - 4 \cdot 2^2 8 + 12 \cdot 1^2 28 \\ - 6 \cdot 156 - 6 \cdot 2^3 6 + 18 \cdot 1^2 2^2 6 \\ - 3 \cdot 25^2 + 9 \cdot 1^2 5^2 - 18 \cdot 2^2 35 + 36 \cdot 1^2 235 + 60 \cdot 12^3 5 - 90 \cdot 1^3 2^2 5 - 2^4 4 - 4 \cdot 2^3 3^2 \\ + 55 \cdot 12^4 3 - 60 \cdot 1^3 2^3 3 + 10 \cdot 2^6 - 105 \cdot 1^2 2^5 + 105 \cdot 1^4 2^4,$$

$$f(123^2) \stackrel{H}{=} 8 - 17 - 4 \cdot 125 + 3 \cdot 1^3 5 - 2 \cdot 134 - 2 \cdot 2^2 4 + 6 \cdot 1^2 24 \\ + 13 \cdot 1^2 2^3 - 1^2 6 - 15 \cdot 1^4 2^2 \\ + 6 \cdot 1^3 23 - 3 \cdot 12^2 3 - 2^4,$$

$$f(1^2 34) \stackrel{H}{=} 8 - 2 \cdot 17 - 26 + 2 \cdot 1^2 6 - 125 - 2 \cdot 2^2 4 - 3 \cdot 134 \\ + 6 \cdot 1^2 24 - 3 \cdot 23^2 + 6 \cdot 1^2 3^2 \\ + 9 \cdot 12^2 3 - 12 \cdot 1^3 23,$$

$$f(1^2 3^3) \stackrel{H}{=} (10) - 2 \cdot 19 - 28 + 2 \cdot 1^2 8 - 3 \cdot 127 - 6 \cdot 136 \\ - 6 \cdot 2^2 6 + 18 \cdot 1^2 26 - 3 \cdot 145 - 9 \cdot 235 \\ + 18 \cdot 1^2 35 + 27 \cdot 12^2 5 - 36 \cdot 1^3 25 - 4 \cdot 2^3 4 + 9 \cdot 1^2 2^2 4 - 12 \cdot 2^2 3^2 \\ + 18 \cdot 1^2 23^2 + 74 \cdot 12^3 3 \\ - 90 \cdot 1^3 2^2 3 + 10 \cdot 2^5 - 95 \cdot 1^2 2^4 + 90 \cdot 1^4 2^3,$$

$$f(1^2 23) \stackrel{H}{=} 6 - 2 \cdot 15 - 24 + 1^2 4 - 6 \cdot 123 + 6 \cdot 1^3 3 - 2 \cdot 2^3 + 15 \cdot 1^2 2^2 - 12 \cdot 1^4 2,$$

$$f(1^3 4) \stackrel{H}{=} 6 - 3 \cdot 15 - 3 \cdot 24 + 6 \cdot 1^2 4 - 3^2 + 6 \cdot 123 - 6 \cdot 1^3 3,$$

$$f(1^3 3^2) \stackrel{H}{=} 8 - 3 \cdot 17 - 3 \cdot 26 + 6 \cdot 1^2 6 - 35 + 4 \cdot 125 - 6 \cdot 1^3 5 \\ - 6 \cdot 134 - 7 \cdot 2^2 4 + 18 \cdot 1^2 24 \\ - 12 \cdot 23^2 + 18 \cdot 1^2 3^2 + 66 \cdot 12^2 3 - 72 \cdot 1^3 23 + 9 \cdot 2^4 - 72 \cdot 1^2 2^3 + 60 \cdot 1^4 2^2,$$

$$f(1^4 3) \stackrel{H}{=} 6 - 4 \cdot 15 - 7 \cdot 24 + 12 \cdot 1^2 4 - 4 \cdot 3^2 + 32 \cdot 123 \\ - 24 \cdot 1^3 3 + 6 \cdot 2^3 - 36 \cdot 1^2 2^2 + 24 \cdot 1^4 2.$$

We skip the extra twenty five terms needed for f_6 when $l_3 \neq 0$. The extra terms needed for $g_r(x, L)$ in (1.34) when $l_3 \neq 0$ are as follows:

$r = 2 :$

$$g(3^2) = H_5 - 2H_2H_3 + H_1H_2^2.$$

$r = 3 :$

$$g(34) = H_6 - H_2H_4 - H_3^2 + H_1H_2H_3,$$

$$g(3^3) = H_8 - 3H_2H_6 - 3H_3H_5 + 3H_1H_2H_5 + 3H_2^2H_4 + 6H_2H_3^2 \\ - 9H_1H_2^2H_3 - H_2^4 + 3H_1^2H_2^3,$$

$$g(23) = H_4 - H_1H_3 - H_2^2 + H_1^2H_2.$$

$r = 4 :$

$$g(35) = 7 - 25 - 34 + 124,$$

$$g(3^24) \stackrel{H}{=} 9 - 2 \cdot 27 - 3 \cdot 36 + 2 \cdot 126 - 45 + 135 + 2^25 + 6 \cdot 234$$

$$+ 2 \cdot 3^3 - 6 \cdot 123^2 - 2^33$$

$$- 3 \cdot 12^24 + 3 \cdot 1^22^23,$$

$$g(3^4) \stackrel{H}{=} 11 - 4 \cdot 29 - 4 \cdot 38 + 4 \cdot 128 + 6 \cdot 2^27 - 6 \cdot 56 + 24 \cdot 236 - 18 \cdot 12^26 + 3 \cdot 15^2$$

$$+ 12 \cdot 245 + 12 \cdot 3^25 - 36 \cdot 1235 - 10 \cdot 2^35 + 18 \cdot 1^22^25 - 36 \cdot 2^234 + 24 \cdot 12^34$$

$$- 24 \cdot 23^3 + 72 \cdot 12^23^2 + 17 \cdot 2^43 - 60 \cdot 1^22^33 - 10 \cdot 12^5 + 15 \cdot 1^32^4,$$

$$g(23^2) \stackrel{H}{=} 7 - 16 - 3 \cdot 25 + 1^25 - 2 \cdot 34 + 4 \cdot 124 + 2 \cdot 13^2 + 5 \cdot 2^23$$

$$- 6 \cdot 1^223 - 4 \cdot 12^3 + 3 \cdot 1^32^2.$$

$r = 5 :$

$$\begin{aligned}
g(36) &\stackrel{H}{=} 8 - 26 - 35 + 125, \\
g(34^2) &\stackrel{H}{=} 10 - 28 - 3 \cdot 37 + 127 - 2 \cdot 46 + 2 \cdot 136 + 2 \cdot 235 + 2 \cdot 24^2 \\
&\quad + 5 \cdot 3^2 4 - 6 \cdot 1234 - 3 \cdot 13^3 - 2^2 3^2 + 3 \cdot 1^2 23^2, \\
g(3^2 5) &\stackrel{H}{=} 10 - 2 \cdot 28 - 2 \cdot 37 + 2 \cdot 127 - 46 + 2^2 6 - 5^2 + 145 + 4 \cdot 235 - 3 \cdot 12^2 5 \\
&\quad + 2 \cdot 24^2 + 2 \cdot 3^2 4 - 6 \cdot 1234 - 2^3 4 + 3 \cdot 1^2 2^2 4, \\
g(3^3 4) &\stackrel{H}{=} (12) - 3 \cdot 2(10) - 4 \cdot 39 + 3 \cdot 129 - 48 + 138 + 3 \cdot 2^2 8 \\
&\quad - 3 \cdot 57 + 15 \cdot 237 - 9 \cdot 12^2 7 \\
&\quad - 3 \cdot 6^2 + 3 \cdot 156 + 12 \cdot 246 + 12 \cdot 3^2 6 - 4 \cdot 2^3 6 - 27 \cdot 1236 + 9 \cdot 1^2 2^2 6 \\
&\quad + 3 \cdot 25^2 + 9 \cdot 345 - 9 \cdot 1245 - 9 \cdot 13^2 5 - 15 \cdot 2^2 35 + 9 \cdot 1^2 235 + 6 \cdot 12^3 5 \\
&\quad - 9 \cdot 2^4 2 - 36 \cdot 23^2 4 + 54 \cdot 12^2 34 + 4 \cdot 2^4 4 - 15 \cdot 1^2 2^3 4 \\
&\quad - 6 \cdot 3^4 + 36 \cdot 123^3 + 13 \cdot 2^3 3^2 - 45 \cdot 1^2 2^2 3^2 - 10 \cdot 12^4 3 + 15 \cdot 1^3 2^3 3, \\
g(3^5) &\stackrel{H}{=} (14) - 5 \cdot 2(12) - 5 \cdot 3(11) + 5 \cdot 12(11) + 10 \cdot 2^2(10) \\
&\quad - 10 \cdot 59 + 40 \cdot 239 - 30 \cdot 12^2 9 \\
&\quad - 10 \cdot 68 + 10 \cdot 158 + 20 \cdot 248 + 20 \cdot 3^2 8 - 60 \cdot 1238 - 20 \cdot 2^3 8 + 30 \cdot 1^2 2^2 8 \\
&\quad + 30 \cdot 257 - 90 \cdot 2^2 37 + 60 \cdot 12^3 7 \\
&\quad + 30 \cdot 26^2 + 60 \cdot 356 - 90 \cdot 1256 - 90 \cdot 2^2 46 - 180 \cdot 23^2 6 + 360 \cdot 12^2 36 \\
&\quad + 45 \cdot 2^4 6 - 150 \cdot 1^2 2^3 6 \\
&\quad + 15 \cdot 45^2 - 60 \cdot 3^3 5 - 45 \cdot 135^2 - 45 \cdot 2^2 5^2 + 45 \cdot 1^2 25^2 - 180 \cdot 2345 + 180 \cdot 12^2 45 \\
&\quad + 360 \cdot 123^2 5 + 210 \cdot 2^3 35 - 450 \cdot 1^2 2^2 35 - 150 \cdot 12^4 5 + 150 \cdot 1^3 2^3 5 \\
&\quad + 60 \cdot 2^3 4^2 + 360 \cdot 2^2 3^2 4 - 600 \cdot 12^3 34 - 51 \cdot 2^5 4 + 225 \cdot 1^2 2^4 4 \\
&\quad + 120 \cdot 23^4 - 600 \cdot 12^2 3^3 - 225 \cdot 2^4 3^2 + 900 \cdot 1^2 2^3 3^2 + 315 \cdot 12^5 3 - 525 \cdot 1^3 2^4 3 \\
&\quad + 10 \cdot 2^7 - 105 \cdot 1^2 2^6 + 105 \cdot 1^4 2^5, \\
g(234) &\stackrel{H}{=} 8 - 17 - 2 \cdot 26 + 1^2 6 - 2 \cdot 35 + 2 \cdot 125 - 4^2 + 4 \cdot 134 + 2 \cdot 2^2 4 - 3 \cdot 1^2 24 \\
&\quad + 3 \cdot 23^2 - 3 \cdot 1^2 3^2 - 4 \cdot 12^2 3 + 3 \cdot 1^3 23, \\
g(23^3) &\stackrel{H}{=} (10) - 19 - 4 \cdot 28 + 1^2 8 - 3 \cdot 37 + 6 \cdot 127 - 3 \cdot 46 \\
&\quad + 6 \cdot 136 + 9 \cdot 2^2 6 - 9 \cdot 1^2 26 \\
&\quad - 3 \cdot 5^2 + 6 \cdot 145 + 21 \cdot 235 - 9 \cdot 1^2 35 - 24 \cdot 12^2 5 + 9 \cdot 1^3 25 + 6 \cdot 24^2 + 6 \cdot 3^2 4 - 36 \cdot 1234 \\
&\quad - 13 \cdot 2^3 4 + 27 \cdot 1^2 2^2 4 - 6 \cdot 13^3 - 27 \cdot 2^2 3^2 + 36 \cdot 1^2 23^2 + 55 \cdot 12^3 3 - 45 \cdot 1^3 2^2 3 \\
&\quad + 4 \cdot 2^5 - 25 \cdot 1^2 2^4 + 15 \cdot 1^4 2^3, \\
g(2^2 3) &\stackrel{H}{=} 6 - 2 \cdot 15 - 3 \cdot 24 + 3 \cdot 1^2 4 - 3^2 + 7 \cdot 123 \\
&\quad - 3 \cdot 1^3 3 + 2 \cdot 2^3 - 7 \cdot 1^2 2^2 + 3 \cdot 1^4 2.
\end{aligned}$$

We shall not give the expressions for the 11 extra (π) needed for g_6 when $l_3 \neq 0$.

Appendix D: Expansions of f_r, g_r in terms of \mathbf{a}

Here, we give the coefficients $f(\pi), g(\pi)$ needed in (1.34) for $f_r, g_r, r \leq 4$ or 5 in terms of \mathbf{a} of (1.26). Again we first give these for the case $l_3 = 0$.

For $r = 2, 3, 4$, $f(\pi)(\mathbf{H})$ has fewer/the same/greater number of terms than $f(\pi)(\mathbf{a})$ in 19/4/5 cases, and $g(\pi)(\mathbf{H})$ has fewer/the same/greater number of terms than $g(\pi)(\mathbf{a})$ in

11/2/2 cases. On the other hand, as a function of x , a_k is generally much briefer than H_k . Here, we give the coefficients needed when $l_3 = 0$ up to $r = 5$. We again adapt the notation of Comtet (1974), but this time in terms of \mathbf{a} , not \mathbf{H} : $k \cdot 1^{i_1} 2^{i_2} \dots \stackrel{a}{=} k a_1^{i_1} a_2^{i_2} \dots$. For example, $15 \cdot 12^2 4^3 \stackrel{a}{=} 15 a_1 a_2^2 a_4^3$. MAPLE gave $f(\pi)$ in terms of \mathbf{a} as follows:

$$\begin{aligned}
r = 2 : f(4) &= H_3 = a_1^3 - 3a_1 a_2 + a_3, \quad f(2) = H_1 = a_1. \\
r = 3 : f(5) &= H_4 \text{ is given by (B.3),} \\
f(14) &= -3a_1^2 a_2 + 3a_2^2 + 3a_1 a_3 - a_4, \\
f(12) &= -a_2. \\
r = 4 : f(6) &= H_5 \text{ is given by (B.4),} \\
f(4^2) &= -15a_1^5 a_2 + 96a_1^3 a_2^2 - 105a_1 a_2^3 \\
&\quad - 204a_1^2 a_2 a_3 + 105a_2^2 a_3 + 33a_1^4 a_3 + 69a_1 a_3^2 \\
&\quad - 35a_1^3 a_4 + 105a_1 a_2 a_4 - 35a_3 a_4 + 21a_1^2 a_5 - 21a_2 a_5 - 7a_1 a_6 + a_7, \\
f(24) &= -7a_1^3 a_2 + 15a_1 a_2^2 + 9a_1^2 a_3 - 10a_2 a_3 - 5a_1 a_4 + a_5, \\
f(2^2) &= -3a_1 a_2 + a_3, \\
f(15) &= -4a_1^3 a_2 + 12a_1 a_2^2 + 6a_1^2 a_3 - 10a_2 a_3 - 4a_1 a_4 + a_5, \\
f(1^2 4) &= 6a_1 a_2^2 + 3a_1^2 a_3 - 9a_2 a_3 - 3a_1 a_4 + a_5, \\
f(1^2 2) &= a_3.
\end{aligned}$$

For f_5 when $l_3 = 0$, see Appendix D.

When $l_3 = 0$, the coefficients $g(\pi)$ in terms of \mathbf{a} needed for $g_r(x)$ are as follows:

$$\begin{aligned}
r = 2 : g(4) &= H_3 \stackrel{a}{=} 1^3 - 3 \cdot 12 + 3, \quad g(2) = H_1 = a_1 \stackrel{a}{=} 1. \\
r = 3 : g(5) &= H_4 \text{ of (B.3).} \\
r = 4 : g(6) &= H_5 \text{ of (B.4),} \\
g(4^2) &\stackrel{a}{=} -9 \cdot 1^5 2 + 72 \cdot 1^3 2^2 - 87 \cdot 12^3 \\
&\quad + 27 \cdot 1^4 3 - 180 \cdot 1^2 23 + 99 \cdot 2^2 3 + 63 \cdot 13^2 + 99 \cdot 124 \\
&\quad - 33 \cdot 34 - 33 \cdot 1^3 4 + 21 \cdot 1^2 5 - 21 \cdot 25 - 7 \cdot 16 + 7, \\
g(24) &\stackrel{a}{=} -3 \cdot 1^3 2 + 9 \cdot 12^2 + 6 \cdot 1^2 3 - 9 \cdot 23 - 4 \cdot 14 + 5, \\
g(2^2) &\stackrel{a}{=} -12 + 3. \\
r = 5 : g(7) &= H_6 \text{ of (B.5),} \\
g(45) &\stackrel{a}{=} -12 \cdot 1^6 2 + 144 \cdot 1^4 2^2 - 348 \cdot 1^2 2^3 + 96 \cdot 2^4 \\
&\quad + 42 \cdot 1^5 3 - 480 \cdot 1^3 23 + 774 \cdot 12^2 3 \\
&\quad + 258 \cdot 1^2 3^2 - 270 \cdot 23^2 \\
&\quad - 64 \cdot 1^4 4 + 396 \cdot 1^2 24 - 204 \cdot 2^2 4 - 268 \cdot 134 + 34 \cdot 4^2 + 55 \cdot 1^3 5 \\
&\quad - 165 \cdot 125 + 55 \cdot 35 - 28 \cdot 1^2 6 + 28 \cdot 26 + 8 \cdot 17 - 8, \\
g(25) &\stackrel{a}{=} -4 \cdot 1^4 2 + 24 \cdot 1^2 2^2 - 12 \cdot 2^3 + 10 \cdot 1^3 3 \\
&\quad - 46 \cdot 123 + 10 \cdot 3^2 - 10 \cdot 1^2 4 + 14 \cdot 24 + 5 \cdot 15 - 6.
\end{aligned}$$

We skip g_6 . The extra terms needed for $f_2, \dots, f_5, g_2, \dots, g_5$ when $l_3 \neq 3$ are given in Appendix D. As functions of \mathbf{a} , the c_r needed for f_r in (4.1) are

$$\begin{aligned}
c_2 &= a_1, \quad c_3 = 2a_1^2 + a_2, \quad c_4 = 3!a_1^3 + 7a_1 a_2 + a_3, \\
c_5 &= 4!a_1^4 + 46a_1^2 a_2 + 11a_1 a_3 + 7a_2^2 + a_4, \\
c_6 &= 5!a_1^5 + 324a_1^3 a_2 + 147a_1^2 a_3 + 127a_1 a_2^2 + 16a_1 a_4 + 25a_2 a_3 + a_5.
\end{aligned}$$

So, c_k has the same number of terms as a function of \mathbf{H} or of \mathbf{a} .

We now give the extra terms needed when $l_3 \neq 0$:

$r = 2$:

$$f(3^2) \stackrel{a}{=} 14 \cdot 12^2 - 8 \cdot 1^3 2 + 10 \cdot 1^2 3 - 10 \cdot 23 - 5 \cdot 14 + 5,$$

$$f(13) \stackrel{a}{=} -2 \cdot 12 + 3.$$

$r = 3$:

$$f(34) \stackrel{a}{=} -11 \cdot 1^4 2 + 42 \cdot 1^2 2^2 - 15 \cdot 2^3 + 19 \cdot 1^3 3 - 59 \cdot 123$$

$$+ 10 \cdot 3^2 - 15 \cdot 1^2 4 + 15 \cdot 24 + 2 \cdot 15 - 6,$$

$$f(3^3) \stackrel{a}{=} 138 \cdot 1^4 2^2 - 374 \cdot 1^2 2^3 + 104 \cdot 2^4$$

$$+ 26 \cdot 1^5 3 - 500 \cdot 1^3 23 + 810 \cdot 12^2 3 + 280 \cdot 1^2 3^2 - 280 \cdot 23^2$$

$$- 280 \cdot 134 - 55 \cdot 1^4 4 + 405 \cdot 1^2 24 - 210 \cdot 2^2 4 + 35 \cdot 4^2$$

$$+ 53 \cdot 1^3 5 - 165 \cdot 125 + 56 \cdot 35$$

$$- 28 \cdot 1^2 6 + 28 \cdot 26 + 8 \cdot 17 - 8,$$

$$f(23) \stackrel{a}{=} -5 \cdot 1^2 2 + 3 \cdot 2^2 + 4 \cdot 13 - 4,$$

$$f(13^2) \stackrel{a}{=} -14 \cdot 2^3 + 24 \cdot 1^2 2^2 + 8 \cdot 1^3 3$$

$$- 48 \cdot 123 + 10 \cdot 3^2 - 10 \cdot 1^2 4 + 15 \cdot 24 + 15 - 6,$$

$$f(1^2 3) \stackrel{a}{=} 2 \cdot 2^2 + 2 \cdot 13 - 4.$$

$r = 4 :$

$$\begin{aligned}
f(35) &\stackrel{a}{=} -14 \cdot 1^5 2 + 96 \cdot 1^3 2^2 - 102 \cdot 12^3 \\
&+ 31 \cdot 1^4 3 - 206 \cdot 1^2 23 + 105 \cdot 2^2 3 + 70 \cdot 13^2 - 102 \cdot 3^3 \\
&+ 104 \cdot 124 - 34 \cdot 1^3 4 - 35 \cdot 34 + 21 \cdot 1^2 5 - 21 \cdot 25 - 7 \cdot 16 + 7, \\
f(3^2 4) &\stackrel{a}{=} 222 \cdot 1^5 2^2 - 1094 \cdot 1^3 2^3 + 912 \cdot 12^4 \\
&- 1053 \cdot 1^4 23 + 35 \cdot 1^6 3 + 3615 \cdot 1^2 2^2 3 \\
&- 2490 \cdot 123^2 + 810 \cdot 1^3 3^2 - 1259 \cdot 2^3 3 + 280 \cdot 3^3 \\
&- 91 \cdot 1^5 4 + 1185 \cdot 1^3 24 - 1860 \cdot 12^2 4 - 1255 \cdot 1^2 34 + 1260 \cdot 234 + 315 \cdot 14^2 \\
&+ 121 \cdot 1^4 5 - 749 \cdot 1^2 25 + 503 \cdot 135 + 378 \cdot 2^2 5 - 126 \cdot 45 \\
&- 82 \cdot 1^3 6 + 250 \cdot 126 - 84 \cdot 36 + 36 \cdot 1^2 7 - 36 \cdot 27 - 9 \cdot 18 + 9, \\
f(3^4) &\stackrel{a}{=} -3792 \cdot 1^5 2^3 + 14512 \cdot 1^3 2^4 - 9892 \cdot 12^5 \\
&- 1792 \cdot 1^6 23 + 27724 \cdot 1^4 2^2 3 + 3200 \cdot 1^5 3^2 \\
&- 43360 \cdot 1^3 23^2 - 64976 \cdot 1^2 2^3 3 + 15400 \cdot 1^2 3^3 + 67910 \cdot 12^2 3^2 \\
&+ 17264 \cdot 2^4 3 - 15400 \cdot 23^3 \\
&- 80 \cdot 1^7 4 - 31680 \cdot 1^3 2^2 4 + 4760 \cdot 1^5 24 - 10130 \cdot 1^4 34 \\
&+ 67880 \cdot 1^2 234 + 5560 \cdot 1^3 4^2 - 17185 \cdot 124^2 + 33780 \cdot 12^3 4 - 23100 \cdot 13^2 4 - 34650 \cdot 2^2 34 \\
&+ 5775 \cdot 34^2 + 244 \cdot 1^6 5 - 5992 \cdot 1^4 25 + 20028 \cdot 1^2 2^2 5 - 27436 \cdot 1235 + 8956 \cdot 1^3 35 \\
&- 6900 \cdot 1^2 45 - 6924 \cdot 2^3 5 + 6930 \cdot 245 + 4620 \cdot 3^2 5 + 1383 \cdot 15^2 \\
&- 350 \cdot 1^5 6 + 4396 \cdot 1^3 26 - 6818 \cdot 12^2 6 - 4620 \cdot 1^2 36 + 2310 \cdot 146 + 4620 \cdot 236 - 462 \cdot 56 \\
&+ 298 \cdot 1^4 7 - 1980 \cdot 1^3 27 + 32 \cdot 1^2 27 + 1320 \cdot 137 + 990 \cdot 2^2 7 - 330 \cdot 47 \\
&+ 491 \cdot 128 - 161 \cdot 1^3 8 - 165 \cdot 38 + 55 \cdot 1^2 9 - 55 \cdot 29 \\
&- 11 \cdot 1(10) + (11), \\
f(23^2) &\stackrel{a}{=} 72 \cdot 1^3 2^2 - 98 \cdot 12^3 \\
&+ 17 \cdot 1^4 3 - 192 \cdot 1^2 23 + 70 \cdot 13^2 + 105 \cdot 2^2 3 - 28 \cdot 1^3 4 \\
&+ 103 \cdot 124 - 35 \cdot 34 + 20 \cdot 1^2 5 - 21 \cdot 25 - 7 \cdot 16 + 7, \\
f(134) &\stackrel{a}{=} -84 \cdot 12^3 - 141 \cdot 1^2 23 + 44 \cdot 1^3 2^2 + 11 \cdot 1^4 3 + 59 \cdot 13^2 + 104 \cdot 2^2 3 - 19 \cdot 1^3 4 \\
&+ 89 \cdot 124 - 35 \cdot 34 + 19 \cdot 1^2 5 - 21 \cdot 25 - 6 \cdot 16 + 7, \\
f(13^3) &\stackrel{a}{=} -552 \cdot 1^3 2^3 + 748 \cdot 12^4 \\
&- 406 \cdot 1^4 23 + 2622 \cdot 1^2 2^2 3 + 500 \cdot 1^3 3^2 - 2180 \cdot 123^2 \\
&- 1226 \cdot 2^3 3 + 280 \cdot 3^3 - 26 \cdot 1^5 4 + 720 \cdot 1^3 24 - 965 \cdot 1^2 34 - 1620 \cdot 12^2 4 + 280 \cdot 14^2 \\
&+ 1260 \cdot 234 + 67 \cdot 1^4 5 - 576 \cdot 1^2 25 + 375 \cdot 2^2 5 + 445 \cdot 135 - 126 \cdot 45 - 53 \cdot 1^3 6 \\
&+ 221 \cdot 126 - 84 \cdot 36 + 28 \cdot 1^2 7 - 36 \cdot 27 - 8 \cdot 18 + 9, \\
f(123) &\stackrel{a}{=} 10 \cdot 12^2 + 5 \cdot 1^2 3 - 10 \cdot 23 - 4 \cdot 14 + 5, \\
f(1^2 3^2) &\stackrel{a}{=} -72 \cdot 1^2 23 - 48 \cdot 12^3 + 48 \cdot 13^2 \\
&+ 90 \cdot 2^2 3 - 8 \cdot 1^3 4 + 68 \cdot 124 - 35 \cdot 34 + 18 \cdot 1^2 5 \\
&- 20 \cdot 25 - 5 \cdot 16 + 7, \\
f(1^3 3) &\stackrel{a}{=} -6 \cdot 23 - 2 \cdot 14 + 5.
\end{aligned}$$

We now give f_5 , but *only when* $l_3 = 0$, as the general case is too long to include here:

$$\begin{aligned}
r = 5 : \quad & f(7) = H_6 \text{ is given by (B.5),} \\
& f(45) \stackrel{a}{=} -19 \cdot 1^6 2 + 189 \cdot 1^4 2^2 - 411 \cdot 1^2 2^3 + 105 \cdot 2^4 \\
& + 51 \cdot 1^5 3 - 542 \cdot 1^3 23 + 276 \cdot 1^2 3^2 + 837 \cdot 12^2 3 - 280 \cdot 23^2 \\
& - 69 \cdot 1^4 4 + 417 \cdot 1^2 24 - 210 \cdot 2^2 4 - 279 \cdot 134 + 35 \cdot 4^2 \\
& - 168 \cdot 125 + 56 \cdot 1^3 5 + 56 \cdot 35 - 28 \cdot 1^2 6 + 28 \cdot 26 + 8 \cdot 17 - 8, \\
& f(25) \stackrel{a}{=} -9 \cdot 1^4 2 + 42 \cdot 1^2 2^2 - 15 \cdot 2^3 + 16 \cdot 1^3 3 \\
& - 60 \cdot 123 + 10 \cdot 3^2 - 14 \cdot 1^2 4 + 15 \cdot 24 + 2 \cdot 15 - 6, \\
& f(16) \stackrel{a}{=} -5 \cdot 1^4 2 + 30 \cdot 1^2 2^2 - 15 \cdot 2^3 + 10 \cdot 1^3 3 \\
& - 50 \cdot 123 + 10 \cdot 3^2 - 10 \cdot 1^2 4 + 15 \cdot 24 + 15 - 6, \\
& f(14^2) \stackrel{a}{=} 75 \cdot 1^4 2^2 - 288 \cdot 1^2 2^3 + 105 \cdot 2^4 \\
& + 15 \cdot 1^5 3 - 324 \cdot 1^3 23 - 33 \cdot 1^4 4 + 723 \cdot 12^2 3 + 204 \cdot 1^2 3^2 \\
& - 279 \cdot 23^2 + 309 \cdot 1^2 24 - 210 \cdot 2^2 4 - 243 \cdot 134 + 35 \cdot 4^2 \\
& + 35 \cdot 1^3 5 - 147 \cdot 125 + 56 \cdot 35 \\
& - 21 \cdot 1^2 6 + 28 \cdot 26 + 7 \cdot 17 - 8, \\
& f(124) \stackrel{a}{=} 21 \cdot 1^2 2^2 - 15 \cdot 2^3 + 7 \cdot 1^3 3 - 48 \cdot 123 \\
& + 10 \cdot 3^2 - 9 \cdot 1^2 4 + 15 \cdot 24 + 15 - 6, \\
& f(12^2) \stackrel{a}{=} 3 \cdot 2^2 + 3 \cdot 13 - 4, \\
& f(1^2 5) \stackrel{a}{=} 12 \cdot 1^2 2^2 - 12 \cdot 2^3 + 4 \cdot 1^3 3 - 36 \cdot 123 + 10 \cdot 3^2 - 6 \cdot 1^2 4 + 14 \cdot 24 - 6, \\
& f(1^3 4) \stackrel{a}{=} -6 \cdot 2^3 - 18 \cdot 123 + 9 \cdot 3^2 - 3 \cdot 1^2 4 + 12 \cdot 24 - 15 - 6, \\
& f(1^3 2) \stackrel{a}{=} -4.
\end{aligned}$$

We skip f_6 . The extra $g(\pi)$ needed for g_r , $r = 2, 3, 4$ in terms of \mathbf{a} as follows:

$$\begin{aligned}
r = 2 : \\
& g(3^2) \stackrel{a}{=} -4 \cdot 1^3 2 + 10 \cdot 12^2 + 8 \cdot 1^2 3 - 8 \cdot 23 - 5 \cdot 14 + 5. \\
r = 3 : \\
& g(34) \stackrel{a}{=} -6 \cdot 1^4 2 + 30 \cdot 1^2 2^2 - 12 \cdot 2^3 + 15 \cdot 1^3 3 \\
& - 51 \cdot 123 + 9 \cdot 3^2 + 14 \cdot 24 - 14 \cdot 1^2 4 + 6 \cdot 15 - 6, \\
& g(3^3) \stackrel{a}{=} 48 \cdot 1^4 2^2 - 212 \cdot 1^2 2^3 + 68 \cdot 2^4 \\
& - 284 \cdot 1^3 23 + 8 \cdot 1^5 3 + 594 \cdot 12^2 3 + 226 \cdot 1^2 3^2 - 226 \cdot 23^2 \\
& - 28 \cdot 1^4 4 + 306 \cdot 1^2 24 - 168 \cdot 2^2 4 - 265 \cdot 134 + 35 \cdot 4^2 \\
& - 144 \cdot 125 + 38 \cdot 1^3 5 + 53 \cdot 35 \\
& - 25 \cdot 1^2 6 + 25 \cdot 26 + 8 \cdot 17 - 8, \\
& g(23) \stackrel{a}{=} -2 \cdot 1^2 2 + 2 \cdot 2^2 + 3 \cdot 13 - 4.
\end{aligned}$$

$r = 4 :$

$$\begin{aligned}
g(35) &\stackrel{a}{=} -8 \cdot 1^5 2 + 68 \cdot 1^3 2^2 - 84 \cdot 12^3 \\
&+ 24 \cdot 1^4 3 - 176 \cdot 1^2 23 + 66 \cdot 13^2 + 92 \cdot 2^2 3 - 30 \cdot 1^3 4 \\
&+ 98 \cdot 124 - 34 \cdot 34 + 20 \cdot 1^2 5 - 20 \cdot 25 - 7 \cdot 16 + 7, \\
g(3^2 4) &\stackrel{a}{=} 84 \cdot 1^5 2^2 - 618 \cdot 1^3 2^3 + 642 \cdot 12^4 \\
&- 600 \cdot 1^4 23 + 2652 \cdot 1^2 2^2 3 + 12 \cdot 1^6 3 \\
&- 2136 \cdot 123^2 - 1002 \cdot 2^3 3 + 624 \cdot 1^3 3^2 + 252 \cdot 3^3 \\
&- 48 \cdot 1^5 4 + 863 \cdot 1^3 24 - 1565 \cdot 12^2 4 \\
&- 1126 \cdot 1^2 34 + 1141 \cdot 234 + 310 \cdot 14^2 + 79 \cdot 1^4 5 - 629 \cdot 1^2 25 + 334 \cdot 2^2 5 + 483 \cdot 135 - 125 \cdot 45 \\
&- 69 \cdot 1^3 6 + 231 \cdot 126 - 81 \cdot 36 + 34 \cdot 1^2 7 - 34 \cdot 27 - 9 \cdot 18 + 9, \\
g(3^4) &\stackrel{a}{=} -1008 \cdot 1^5 2^3 + 6752 \cdot 1^3 2^4 - 6044 \cdot 12^5 \\
&- 416 \cdot 1^6 23 + 12224 \cdot 1^4 2^2 3 - 41344 \cdot 1^2 2^3 3 + 12124 \cdot 2^4 3 + 1264 \cdot 1^5 3^2 \\
&- 27728 \cdot 1^3 23^2 + 52552 \cdot 12^2 3^2 + 12896 \cdot 1^2 3^3 - 12896 \cdot 23^3 \\
&- 16 \cdot 1^7 4 + 1840 \cdot 1^5 24 - 19320 \cdot 1^3 2^2 4 - 15395 \cdot 124^2 - 5864 \cdot 1^4 34 + 53728 \cdot 1^2 234 \\
&- 28524 \cdot 2^2 34 - 21740 \cdot 13^2 4 + 25276 \cdot 12^3 4 + 4200 \cdot 1^3 4^2 + 5635 \cdot 34^2 \\
&+ 80 \cdot 1^6 5 - 3320 \cdot 1^4 25 + 14988 \cdot 1^2 2^2 5 - 5480 \cdot 2^3 5 \\
&- 24152 \cdot 1235 + 6760 \cdot 1^3 35 + 4348 \cdot 3^2 5 - 6198 \cdot 1^2 45 + 6348 \cdot 245 + 1353 \cdot 15^2 \\
&- 168 \cdot 1^5 6 + 3136 \cdot 1^3 26 - 5704 \cdot 12^2 6 - 4136 \cdot 1^2 36 \\
&+ 4136 \cdot 236 + 2280 \cdot 146 - 456 \cdot 56 \\
&+ 192 \cdot 1^4 7 - 1640 \cdot 1^2 27 + 852 \cdot 2^2 7 + 1288 \cdot 137 - 330 \cdot 47 \\
&- 129 \cdot 1^3 8 + 451 \cdot 128 - 161 \cdot 38 \\
&+ 51 \cdot 1^2 9 - 51 \cdot 29 - 11 \cdot 1(10) + (11), \\
g(23^2) &\stackrel{a}{=} 20 \cdot 1^3 2^2 - 50 \cdot 12^3 + 4 \cdot 1^4 3 \\
&- 96 \cdot 1^2 23 + 74 \cdot 2^2 3 + 54 \cdot 13^2 \\
&- 12 \cdot 1^3 4 + 73 \cdot 124 - 33 \cdot 34 + 13 \cdot 1^2 5 - 18 \cdot 25 - 6 \cdot 16 + 7.
\end{aligned}$$

Appendix E: Expansions of f_r , g_r for the gamma when $l_3 = 0$

Here, we give the coefficients $f(\pi)$, $g(\pi)$ needed in (1.34) for f_r , g_r , $r \leq 4$ or 5 when $l_3 = 0$ for $X = G$ a gamma variable with mean m in terms of $\alpha = m - 1$, $\bar{y} = -1/y$, as in (2.2).

One can show

$$f(1^i, k+2) = (-1)^i (k+1) \sigma^{k+i+1} \sum_{j=0}^k \binom{k}{j} [\alpha]_{j+1} \bar{y}^{j+i+1} (j+2)_{i-1}, \quad i \geq 1, \quad (\text{E.1})$$

$$f(1^i, 2, k+1) = (-1)^{i-1} \sigma^{k+i+2} \sum_{j=0}^k \binom{k}{j} [\alpha]_j \bar{y}^{j+i+2} \{(2k+1)(\alpha-j) + j^2\} (j+2)_i. \quad (\text{E.2})$$

The other $f(\pi)$ are more simply given by the formulas for them in Appendix C, but we

give them here for the record:

$$\begin{aligned}
r = 2 : & \text{ for } k = 4, 2, f(k) = H_{k-1} \text{ of (2.3).} \\
r = 3 : & f(\pi) \text{ of (E.1) for } \pi = 5, 12, 14. \\
r = 4 : & f(\pi) \text{ of (E.1), (E.2) for } \pi = 6, 15, 24, 2^2, 1^2 4, 1^2 2, 25, \\
& f(4^2) / 3\sigma^7 = -5\alpha \bar{y}^2 - \alpha(25\alpha - 22) \bar{y}^3 - 10[\alpha]_2(5\alpha - 7) \bar{y}^4 \\
& - 2[\alpha]_2(25\alpha^2 - 89\alpha + 84) \bar{y}^5 - [\alpha]_3(25\alpha^2 - 109\alpha + 140) \bar{y}^6 \\
& - [\alpha]_3(5\alpha^3 - 39\alpha^2 + 114\alpha - 120) \bar{y}^7. \\
r = 5 : & f(\pi) \text{ of (E.1), (E.2) for } \pi = 7, 16, 25, 124, 12^2, 1^2 5, 1^3 4, 1^3 2, \\
& f(45) / \sigma^8 = -19\alpha \bar{y}^2 - 6\alpha(19\alpha - 17) \bar{y}^3 - 3[\alpha]_2(95\alpha - 138) \bar{y}^4 \\
& - 4[\alpha]_2(95\alpha^2 - 349\alpha + 336) \bar{y}^5 - 3[\alpha]_3(95\alpha^2 - 433\alpha + 560) \bar{y}^6 \\
& - 6[\alpha]_3(19\alpha^3 - 154\alpha^2 + 455\alpha - 480) \bar{y}^7 - [\alpha]_4(19\alpha^3 - 177\alpha^2 + 638\alpha - 840) \bar{y}^8, \\
& f(14^2) / 3\sigma^8 = 10\alpha \bar{y}^3 + 3\alpha(25\alpha - 22) \bar{y}^4 + 40[\alpha]_2(5\alpha - 7) \bar{y}^5 \\
& + 10[\alpha]_2(25\alpha^2 - 89\alpha + 84) \bar{y}^6 + 6[\alpha]_3(25\alpha^2 - 109\alpha + 140) \bar{y}^7 \\
& + 7[\alpha]_3(5\alpha^3 - 39\alpha^2 + 114\alpha - 120) \bar{y}^8.
\end{aligned}$$

$$\begin{aligned}
r = 6 : \quad & f(\pi) \text{ of (E.1), (E.2) for } \pi = 8, 17, 26, 1^2 6, 1^3 5, 1^4 4, 1^4 2, 125, 1^2 24, 1^2 2^2, \\
& f(5^2)/2\sigma^9 = -12\alpha\overline{y}^2 - 4\alpha(21\alpha - 19)\overline{y}^3 - 12[\alpha]_2(21\alpha - 31)\overline{y}^4 \\
& -12[\alpha]_2(35\alpha^2 - 130\alpha + 126)\overline{y}^5 \\
& -20[\alpha]_3(21\alpha^2 - 97\alpha + 126)\overline{y}^6 - 12[\alpha]_3(21\alpha^3 - 172\alpha^2 + 511\alpha - 540)\overline{y}^7 \\
& -12[\alpha]_4(7\alpha^3 - 66\alpha^2 + 239\alpha - 315)\overline{y}^8 - 4[\alpha]_4(3\alpha^4 - 43\alpha^3 + 258\alpha^2 - 743\alpha + 840)\overline{y}^9, \\
& f(46)/\sigma^9 = -23\alpha\overline{y}^2 - \alpha(161\alpha - 146)\overline{y}^3 - 3[\alpha]_2(161\alpha - 242)\overline{y}^4 \\
& -5[\alpha]_2(161\alpha^2 - 610\alpha + 600)\overline{y}^5 \\
& -5[\alpha]_3(161\alpha^2 - 767\alpha + 1008)\overline{y}^6 \\
& -3[\alpha]_3(161\alpha^3 - 1357\alpha^2 + 4082\alpha - 4320)\overline{y}^7 \\
& -[\alpha]_4(161\alpha^3 - 1575\alpha^2 + 5734\alpha - 7560)\overline{y}^8 \\
& -[\alpha]_5(23\alpha^3 - 249\alpha^2 + 1066\alpha - 1680)\overline{y}^9, \\
& f(4^3)/9\sigma^{11} = 14\alpha\overline{y}^3 + 3\alpha(57\alpha - 50)\overline{y}^4 \\
& +\alpha(805\alpha^2 - 1860\alpha + 1064)\overline{y}^5 \\
& +[\alpha]_2(2023\alpha^2 - 6620\alpha + 5880)\overline{y}^6 \\
& +3[\alpha]_2(1015\alpha^3 - 5840\alpha^2 + 12056\alpha - 8720)\overline{y}^7 \\
& +[\alpha]_2(2849\alpha^4 - 24931\alpha^3 + 87918\alpha^2 - 145044\alpha + 92400)\overline{y}^8 \\
& +[\alpha]_3(1631\alpha^4 - 16737\alpha^3 + 71986\alpha^2 - 149400\alpha + 123200)\overline{y}^9 \\
& +[\alpha]_3(525\alpha^5 - 7500\alpha^4 + 47457\alpha^3 - 162474\alpha^2 + 294472\alpha - 221760)\overline{y}^{10} \\
& +[\alpha]_3(73\alpha^6 - 1374\alpha^5 + 11877\alpha^4 - 59148\alpha^3 + 175372\alpha^2 - 288080\alpha + 201600)\overline{y}^{11}, \\
& f(24^2)/\sigma^9 = 62\alpha\overline{y}^3 + 81\alpha(7\alpha - 6)\overline{y}^4 \\
& +3\alpha(635\alpha^2 - 1450\alpha + 824)\overline{y}^5 \\
& +22[\alpha]_2(145\alpha^2 - 473\alpha + 420)\overline{y}^6 \\
& +36[\alpha]_2(80\alpha^3 - 463\alpha^2 + 961\alpha - 700)\overline{y}^7 \\
& +3[\alpha]_3(449\alpha^3 - 3079\alpha^2 + 8006\alpha - 7560)\overline{y}^8 \\
& +[\alpha]_3(257\alpha^4 - 2691\alpha^3 + 11746\alpha^2 - 24552\alpha + 20160)\overline{y}^9, \\
& f(2^2 4)/3\sigma^7 = 10\alpha\overline{y}^3 + \alpha(61\alpha - 50)\overline{y}^4 \\
& +\alpha(123\alpha^2 - 272\alpha + 152)\overline{y}^5 \\
& +[\alpha]_2(103\alpha^2 - 326\alpha + 280)\overline{y}^6 \\
& +[\alpha]_3(31\alpha^2 - 114\alpha + 120)\overline{y}^7, \\
& f(2^3)/\sigma^5 = 14\alpha\overline{y}^3 + \alpha(43\alpha - 30)\overline{y}^4 \\
& +\alpha(29\alpha^2 - 50\alpha + 24)\overline{y}^5, \\
& f(145)/2\sigma^9 = 19\alpha\overline{y}^3 + 9\alpha(19\alpha - 17)\overline{y}^4 + 6[\alpha]_2(95\alpha - 138)\overline{y}^5 \\
& +10[\alpha]_2(95\alpha^2 - 349\alpha + 336)\overline{y}^6 \\
& +9[\alpha]_3(95\alpha^2 - 433\alpha + 560)\overline{y}^7 \\
& +21[\alpha]_3(19\alpha^3 - 154\alpha^2 + 455\alpha - 480)\overline{y}^8 \\
& +4[\alpha]_4(19\alpha^3 - 177\alpha^2 + 638\alpha - 840)\overline{y}^9, \\
& f(1^2 4^2)/6\sigma^9 = -15\alpha\overline{y}^4 - 6\alpha(25\alpha - 22)\overline{y}^5 - 100[\alpha]_2(5\alpha - 7)\overline{y}^6 \\
& -30[\alpha]_2(25\alpha^2 - 89\alpha + 84)\overline{y}^7 \\
& -21[\alpha]_3(25\alpha^2 - 109\alpha + 140)\overline{y}^8 \\
& -28[\alpha]_3(5\alpha^3 - 39\alpha^2 + 114\alpha - 120)\overline{y}^9.
\end{aligned}$$

When $l_3 = 0$, $g(\pi)$ needed for g_r in (1.34) are as follows:

$$g(2, k+1) = -\sigma^{k+2} \sum_{j=0}^k \binom{k}{j} [\alpha]_{j+1} \{k\alpha - (k-1)j\} \bar{y}^{j+2}. \quad (\text{E.3})$$

$r = 2$: for $k = 4, 2$, $g(k) = H_{k-1}$ of (2.3).

$r = 3$: $g(5) = H_4$ of (2.3).

$r = 4$: $g(6) = H_5$ of (2.3) and $g(24), g(2^2)$ of (E.3),

$$\begin{aligned} g(4^2)/3\sigma^7 &= -3\alpha\bar{y}^2 - 3\alpha(5\alpha - 6)\bar{y}^3 - 6[\alpha]_2(5\alpha - 11)\bar{y}^4 \\ &\quad - 6[\alpha]_2(5\alpha^2 - 25\alpha + 28)\bar{y}^5 - [\alpha]_3(15\alpha^2 - 99\alpha + 140)\bar{y}^6 \\ &\quad - [\alpha]_3(\alpha - 6)(3\alpha^2 - 15\alpha + 20)\bar{y}^7. \end{aligned}$$

$r = 5$: $g(7) = H_6$ of (2.3), $g(25)$ of (E.3),

$$\begin{aligned} g(45)/12\sigma^8 &= -\alpha\bar{y}^2 - \alpha(6\alpha - 7)\bar{y}^3 - [\alpha]_2(15\alpha - 32)\bar{y}^4 \\ &\quad - 2[\alpha]_2(10\alpha^2 - 49\alpha + 55)\bar{y}^5 - [\alpha]_3(15\alpha^2 - 97\alpha + 140)\bar{y}^6 \\ &\quad - [\alpha]_3(6\alpha^3 - 65\alpha^2 + 219\alpha - 240)\bar{y}^7 - [\alpha]_4(\alpha - 7)(\alpha^2 - 6\alpha + 10)\bar{y}^8. \end{aligned}$$

$r = 6$: $g(8) = H_7$ of (2.3), $g(26)$ of (E.3),

$$\begin{aligned} g(5^2)/8\sigma^9 &= -2\alpha\bar{y}^2 - 2\alpha(7\alpha - 8)\bar{y}^3 - 3[\alpha]_2(14\alpha - 29)\bar{y}^4 - [\alpha]_2(70\alpha^2 - 335\alpha + 372)\bar{y}^5 \\ &\quad - 10[\alpha]_3(7\alpha^2 - 44\alpha + 63)\bar{y}^6 - 6[\alpha]_3(7\alpha^3 - 74\alpha^2 + 247\alpha - 270)\bar{y}^7 \\ &\quad - [\alpha]_4(14\alpha^3 - 177\alpha^2 + 703\alpha - 945)\bar{y}^8 - [\alpha]_4(\alpha - 8)(2\alpha - 7)(\alpha^2 - 7\alpha + 15)\bar{y}^9, \\ g(46)/15\sigma^9 &= -\alpha\bar{y}^2 - \alpha(7\alpha - 8)\bar{y}^3 - [\alpha]_2(21\alpha - 44)\bar{y}^4 \\ &\quad - [\alpha]_2(35\alpha^2 - 170\alpha + 192)\bar{y}^5 - [\alpha]_3(35\alpha^2 - 225\alpha + 332)\bar{y}^6 \\ &\quad - [\alpha]_3(21\alpha^3 - 227\alpha^2 + 776\alpha - 864)\bar{y}^7 - [\alpha]_4(7\alpha^3 - 91\alpha^2 + 372\alpha - 504)\bar{y}^8 \\ &\quad - [\alpha]_5(\alpha - 8)(\alpha^2 - 7\alpha + 14)\bar{y}^9, \\ g(4^3)/9\sigma^{11} &= 6\alpha\bar{y}^3 + 12\alpha(5\alpha - 6)\bar{y}^4 + 3\alpha(119\alpha^2 - 380\alpha + 264)\bar{y}^5 \\ &\quad + 3[\alpha]_2(301\alpha^2 - 1480\alpha + 1680)\bar{y}^6 + 3[\alpha]_2(455\alpha^3 - 3800\alpha^2 + 9864\alpha - 8160)\bar{y}^7 \\ &\quad + 3[\alpha]_2(427\alpha^4 - 5333\alpha^3 + 23454\alpha^2 - 44092\alpha + 30240)\bar{y}^8 \\ &\quad + [\alpha]_3(735\alpha^4 - 11265\alpha^3 + 60786\alpha^2 - 141816\alpha + 123200)\bar{y}^9 \\ &\quad + [\alpha]_3(237\alpha^5 - 4944\alpha^4 + 39033\alpha^3 - 150558\alpha^2 + 288712\alpha - 221760)\bar{y}^{10} \\ &\quad + [\alpha]_3(\alpha - 10)(33\alpha^5 - 564\alpha^4 + 3957\alpha^3 - 14298\alpha^2 + 26552\alpha - 20160)\bar{y}^{11}, \\ g(24^2)/3\sigma^9 &= 6\alpha\bar{y}^3 + 3\alpha(19\alpha - 24)\bar{y}^4 + 3\alpha(65\alpha^2 - 222\alpha + 160)\bar{y}^5 \\ &\quad + 6[\alpha]_2(55\alpha^2 - 297\alpha + 360)\bar{y}^6 + 12[\alpha]_2(25\alpha^3 - 230\alpha^2 + 642\alpha - 560)\bar{y}^7 \\ &\quad + [\alpha]_3(141\alpha^3 - 1665\alpha^2 + 5948\alpha - 6720)\bar{y}^8 \\ &\quad + [\alpha]_3(\alpha - 8)(3\alpha - 10)(9\alpha^2 - 53\alpha + 84)\bar{y}^9, \\ g(2^2 4)/3\sigma^7 &= 2\alpha\bar{y}^3 + \alpha(13\alpha - 18)\bar{y}^4 + \alpha(27\alpha^2 - 104\alpha + 80)\bar{y}^5 \\ &\quad + [\alpha]_3(23\alpha - 100)\bar{y}^6 + [\alpha]_3(\alpha - 6)(7\alpha - 20)\bar{y}^7, \\ g(2^3)/\sigma^5 &= 2\alpha\bar{y}^3 + \alpha(7\alpha - 12)\bar{y}^4 + \alpha(\alpha - 4)(5\alpha - 6)\bar{y}^5. \end{aligned}$$

Appendix F: Comparison with Cornish and Fisher

Here, we specialize to the case $X = N$, so that $H_r = He_r$, the r th Hermite polynomial of (1.36). We give $f(\pi)$, $g(\pi)$ of (1.34) needed for f_r , g_r of (1.10) and (1.11). We also give special formulas for some of them that do not hold in the non-normal case. Recall that for $i \geq 2$, $\mathbb{E} N^{2j} = 1 \cdot 3 \cdots (2j - 1)$. Using the notation (1.38), we have

$$f(1^k, k) = 0, \quad f(1^j, k + 1) = (-1)^j [k]_j H_{k-j}, \quad f(2^j) = (-1)^{j-1} H_1 \mathbb{E} N^{2j}.$$

As above, special relations are put in the line starting 'r ='. $f(\pi)$ needed for f_r are as follows:

$$\begin{aligned} r = 1: & \quad f(3) = H_2, \quad f(1) = 1. \\ r = 2: & \quad f(4) = H_3, \quad f(2) = H_1, \quad f(13) = -2H_1, \\ & \quad f(3^2) = -2(4x^3 - 7x). \\ r = 3: & \quad f(5) = H_4, \quad f(14) = -3H_2, \quad f(12) = -1, \quad f(1^2 3) = 2, \\ & \quad f(34) = -11x^4 + 42x^3 - 15, \quad f(3^3) = 2(69x^4 - 187x^3 + 52), \\ & \quad f(23) = -5x^2 + 3, \quad f(13^2) = 2(12x^2 - 7). \\ r = 4: & \quad f(6) = H_5, \quad f(2^2) = -3H_1, \quad f(15) = -4H_3, \quad f(1^2 4) = 6H_1, \\ & \quad f(1^2 2) = f(1^3 3) = 0, \\ & \quad f(4^2) = -3(5x^5 - 32x^3 + 35x), \quad f(35) = -2(7x^5 - 48x^3 + 51x), \\ & \quad f(3^2 4) = 2(111x^5 - 547x^3 + 456x), \quad f(3^4) = -4(948x^5 - 3628x^3 + 2473x), \\ & \quad f(24) = -(7x^3 - 15x), \quad f(23^2) = 2(36x^3 - 49x), \quad f(134) = 4(11x^3 - 21x), \\ & \quad f(13^3) = -4(138x^3 - 187x), \quad f(123) = 10H_1, \quad f(1^2 3^2) = -48H_1. \end{aligned}$$

$$\begin{aligned} r = 5: & \quad f(7) = H_6, \quad f(16) = -5H_4, \quad f(1^2 5) = 12H_2, \quad f(1^3 4) = -6, \\ & \quad f(1^3 2) = f(1^4 3) = 0, \\ & \quad f(45) = -19x^6 + 189x^4 - 411x^2 + 105, \quad f(36) = -17x^6 + 185x^4 - 405x^2 + 105, \\ & \quad f(34^2) = 347x^6 - 2643x^4 + 4521x^2 - 945, \quad f(3^2 5) = 2(162x^6 - 1309x^4 + 2232x^2 - 471), \\ & \quad f(3^3 4) = -2(3354x^6 - 20831x^4 + 29148x^2 - 5174), \\ & \quad f(3^5) = 4(36240x^6 - 184146x^4 + 217921x^2 - 33523), \\ & \quad f(25) = -3(3x^4 - 14x^2 + 5), \quad f(234) = 121x^4 - 378x^2 + 105, \\ & \quad f(23^3) = -2(897x^4 - 2057x^2 + 468), \quad f(2^2 3) = 5(7x^2 - 3), \\ & \quad f(14^2) = 3(25x^4 - 96x^2 + 35), \quad f(135) = 2(35x^4 - 144x^2 + 51), \\ & \quad f(13^2 4) = -6(185x^4 - 547x^2 + 152), \quad f(13^4) = 4(4740x^4 - 10884x^2 + 2473), \\ & \quad f(124) = 3(7x^2 - 5), \quad f(123^2) = -2(108x^2 - 49), \quad f(12^2) = 3, \\ & \quad f(1^2 34) = -12(11x^2 - 7), \quad f(1^2 3^3) = 4(414x^2 - 187), \quad f(1^2 23) = -10, \quad f(1^3 3^2) = 48. \end{aligned}$$

f_6 has forty six terms, so is only given here for the case $l_3 = 0$:

$$\begin{aligned}
r = 6 : f(8) &= H_7, f(17) = -6H_5, f(1^2 6) = 20H_3, f(1^3 5) = -24H_1, f(2^3) = 15H_1, \\
f(1^3 3) &= f(1^4 4) = f(1^5 3) = f(1^4 2) = f(1^4 3^2) = f(1^3 23) = 0, \\
f(5^2) &= -24(x^7 - 14x^5 + 51x^3 - 39x), f(46) = -23x^7 + 333x^5 - 1215x^3 + 945x, \\
f(4^3) &= 3(177x^7 - 1899x^5 + 5451x^3 - 3465x), f(26) = -11x^5 + 90x^3 - 105x, \\
f(24^2) &= 3(65x^5 - 352x^3 + 315x), f(2^2 4) = 21(3x^3 - 5x), \\
f(145) &= 6(19x^5 - 126x^3 + 137x), f(125) = 12(3x^3 - 7x), \\
f(1^2 4^2) &= -12(25x^3 - 48x), f(1^2 24) = -42H_1.
\end{aligned}$$

Note the special relations

$$f(1^2 3^2) = -48H_1, f(1^2 24) = -42H_1, f(123) = 10H_1.$$

For $k \geq 2, i \geq 0, g(2^i, k) = (-1)^i \nu_{ki} H_{k-1}$, where $\nu_{k0} = 1, \nu_{ki} = (k-1)(k+1) \cdots (k+2i-3)$ for $i \geq 1$. That is,

$$g(2^i, r+2-2i) = (-1)^i \langle r \rangle_i H_{r+1-2i}, \quad 0 \leq i \leq r/2,$$

where

$$\begin{aligned}
\langle r \rangle_0 &= 1, \\
\langle r \rangle_i &= (r-1)(r-3) \cdots (r-2i+1) = 2^i \Gamma(s) / \Gamma(s-i), \quad s = (r+1)/2, \quad i \geq 1.
\end{aligned}$$

Again, cases of this relation are put in the line next to ' $r =$ '. The coefficients $g(\pi)$ of (1.34) needed for g_r are as follows:

$$\begin{aligned}
r = 1 : g(3) &= H_2, g(1) = 1. \\
r = 2 : g(4) &= H_3, g(2) = H_1, \\
g(3^2) &= -2(2x^3 - 5x). \\
r = 3 : g(5) &= H_4, g(23) = -2H_2, \\
g(34) &= -6(x^4 - 5x^2 + 2), g(3^3) = 4(12x^4 - 53x^2 + 17). \\
r = 4 : g(6) &= H_5, g(24) = -3H_3, g(2^2) = -H_1, \\
g(4^2) &= -3(3x^5 - 24x^3 + 29x), g(35) = -4(2x^5 - 17x^3 + 21x), \\
g(3^2 4) &= 6(14x^5 - 103x^3 + 107x), g(3^4) = -4(252x^5 - 1688x^3 + 1511x), \\
g(23^2) &= -5g(3^2). \\
r = 5 : g(7) &= H_6, g(25) = -4H_4, g(2^2 3) = 8H_2, \\
g(45) &= -12(x^6 - 12x^4 + 29x^2 - 8), g(36) = -10(x^6 - 13x^4 + 33x^2 - 9), \\
g(34^2) &= 12(12x^6 - 129x^4 + 271x^2 - 64), g(3^2 5) = 8(16x^6 - 181x^4 + 393x^2 - 90), \\
g(3^3 4) &= -24(80x^6 - 803x^4 + 1513x^2 - 304), \\
g(3^5) &= 32(960x^6 - 8937x^4 + 15062x^2 - 2651), \\
g(234) &= -6g(34), g(23^3) = -8g(3^3).
\end{aligned}$$

$$\begin{aligned}
r = 6 : \quad & g(8) = H_7, \quad g(26) = -5H_5, \quad g(2^2 4) = 15H_3, \quad g(2^3) = 3H_1, \\
& g(5^2) = -8(2x^7 - 33x^5 + 132x^3 - 108x), \\
& g(46) = -15(x^7 - 17x^5 + 69x^3 - 57x), \\
& g(4^3) = 27(9x^7 - 131x^5 + 451x^3 - 321x), \\
& g(37) = 6(2x^7 - 37x^5 + 160x^3 - 135x), \\
& g(345) = 12(18x^7 - 273x^5 + 974x^3 - 695x), \\
& g(3^2 6) = 10(18x^7 - 293x^5 + 1100x^3 - 795x), \\
& g(3^2 4^2) = -6(594x^7 - 8193x^5 + 26006x^3 - 16367x), \\
& g(3^3 5) = -8(396x^7 - 5708x^5 + 18755x^3 - 11811x), \\
& g(3^4 4) = 12(5148x^7 - 67004x^5 + 195259x^3 - 109553x), \\
& g(3^6) = -8(154440x^7 - 1887684x^5 + 5033714x^3 - 2542637x), \\
& g(24^2) = -7g(4^2), \quad g(235) = -7g(35), \quad g(23^2 4) = -9g(3^2 4), \\
& g(23^4) = -11g(3^4), \quad g(2^2 3^2) = 35g(3^2).
\end{aligned}$$

These results agree with those given on page 317 of Cornish and Fisher (1937) and page 214 of Fisher and Cornish (1960), except for (i) a typo on page 316: ab was written for ad , making $f(14)$ above into a second $f(12)$; and (ii) they gave $f(15) = -4(x^3 - x)$ instead of $-4H_3$. The factor -6 in $g(34)$ was omitted by Hill and Davis (1968). x in $f(3^4)$ was omitted in Withers (1984). Fisher and Cornish did not give h_r, f_r for $r = 5, 6$. Hill and Davis reference a Stanford Technical Report which may give $e_r(x, L)$ for $e = h, f, g$ up to $r = 11$ when $X = N$.

Note how $g(2\pi)$ is a multiple of $g(\pi)$. For example, if $\pi = (3^{i_3} 4^{i_4} \dots)$, then $g(2\pi) = (1 - |\pi|)g(\pi)$, where $|\pi| = \sum_{k=3} k i_k$.

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Table 1.1 Number of terms needed by $e_r(x)$ for different choices of X .

J, K :		0, 1	1, 2	JK	J	K		01	12	JK	saving
	Number of terms, $N + M$						Cumulative number of terms, $N + M$				
e_0	1+ 0	1+ 0	1+ 0	1+ 0	0	1	1+ 0	1+ 0	1+ 0	1+ 0	0%
e_1	2+ 0	“	0+ 0	0+ 0	1	1	3+ 0	2+ 0	“	“	67%
h_2	5+ 0	3+ 0	1+ 0	1+ 0	1	2	8+ 0	5+ 0	2+ 0	2+ 0	75%
f_2	3+ 0	2+ 0	“	“	1	2	6+ 0	4+ 0	“	“	67%
g_2	“	“	“	“	1	2	“	“	“	“	67%
h_3	9+ 2	4+ 2	1+ 2	1+ 1	2	2	17+ 2	9+ 2	3+ 2	3+ 1	79%
f_3	8+ 2	3+ 2	“	“	2	2	14+ 2	7+ 2	“	“	75%
g_3	4+ 2	1+ 2	“	“	2	2	10+ 2	5+ 2	“	“	67%
h_4	17+ 6	8+ 4	2+ 2	2+ 1	2	3	34+ 8	17+ 6	5+ 4	5+ 2	83%
f_4	14+ 5	7+ 4	“	“	2	3	28+ 7	14+ 6	“	“	80%
g_4	8+ 3	4+ 2	“	“	2	3	18+ 5	9+ 4	“	“	70%
h_5	28+15	11+10	2+ 5	2+ 2	3	3	62+23	28+16	7+ 9	7+ 4	87%
f_5	25+15	10+10	“	“	3	3	53+22	24+16	“	“	85%
g_5	11+ 8	3+ 5	2+ 4	“	3	3	29+13	12+ 9	7+ 8	“	74%
h_6	46+42	19+32	4+10	4+ 3	3	4	108+65	47+48	11+19	11+ 7	90%
f_6	40+37	18+22	4+ 9	“	3	4	93+59	42+38	11+18	“	88%
g_6	19+16	8+ 9	4+ 7	“	3	4	48+29	20+18	11+15	“	77%

Table 2.1 Two approximations for the 95th quantile of $2^{-1} \ln F_{24,60}$.

Order of magnitude	Successive terms	Successive totals	Successive errors
0	.2809 1224	.2809 1224	.0155 6380
1	-.0196 0643	.2613 0581	-.0040 4263
2	.0044 6851	.2657 7432	.0004 2588
3	-.0004 8004	.2652 9428	-.0000 5416
4	.0000 5645	.2653 5073	.0000 0229
5	-.0000 0154	.2653 4919	-.0000 0075
6	-.0000 0102	.2653 4817	.0000 0027

Table 2.2 Two approximations for the 95 quantile of $2^{-1} \ln F_{5,5}$.

Order of magnitude	Successive terms	Successive totals	Successive errors
0	.0	.	.
1	-.0	.	-.00
2	.0	.	.00
3	-.0	.	-.00
4	.0	.	.00
5	-.0	.	-.00
6	-.0	.	.00